Bass's Work in Ring Theory and Projective Modules

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This paper is dedicated to Hyman Bass on his 65th birthday.

ABSTRACT. The early papers of Hyman Bass in the late 50s and the early 60s leading up to his pioneering work in algebraic K-theory have played an important and very special role in ring theory and the theory of projective (and injective) modules. In this article, we give a general survey of Bass's fundamental contributions in this early period of his work, and explain how much this work has influenced and shaped the thinking of subsequent researchers in the area.

Contents

§0. Introduction

It gives me great pleasure to have this opportunity to write about Professor Hyman Bass's work in ring theory and projective modules. This was work done by a young Hyman in the early 60s when he was a junior faculty member at Columbia. The now-classical paper on the homological generalization of semiprimary rings [B_1], an outgrowth of his 1959 Chicago thesis written under the direction of Irving Kaplansky, was followed in quick succession by a brilliant series of papers [B_2–B_7] on various aspects of the structure of projective modules (decomposition, extendibility, and freeness) and injective modules (injective dimensions, Gorenstein rings, and Bass rings). The 1962 announcement of a homotopy theory of projective modules with Schanuel [BS] and the subsequent 1964 announcement [B_8] on the stable structure of the general linear group over an arbitrary ring culminated in his famous IHES paper [B_9] which, perhaps more than any other work in that era, marked the monumental creation of the new mathematical subject of algebraic K-theory. The usual way to put it would be to say, I guess, that all of this work "showed the future master"; but in fact, by this time, Hyman Bass had already proved himself to be a true master of the great art of algebra — and a mathematician of extraordinary creativity and insight.

Since Chuck Weibel [We] will be touching upon Bass's work in algebraic K-theory and Craig Huneke [Hu] will be reporting on his work on injective modules and Gorenstein rings, I shall focus the discussion in this article mainly on Bass's contributions to ring theory and the structure of projective modules. Starting with the latter, we first survey, in §1 and §2 below, Bass's work on big projectives and on his generalizations and extensions of Serre's results on the stable structure of projective modules. In §3, we move on to Bass's work surrounding the theme of Serre's Conjecture on f.g.[1] projective modules over polynomial rings. The last section of Part I concludes with a survey on Bass's work on the decomposition of torsionfree modules and on commutative rings with binary generated ideals, which resulted in the important notion of Bass rings and Bass orders. In this section (§4), Gorenstein rings play a substantial role, but our exposition is designed to overlap only peripherally with that of Huneke's article.

In Part II of this article, we return to Bass's maiden work on the homological generalizations of semiprimary rings, and explain the historic context of this work as well as the role it played in the subsequent development of noncommutative ring theory. Guided by the ideas of stability in the homotopy theory of vector bundles, Bass also single-handedly invented the notion of stable range of a ring R, which he successfully applied to the study of the stable structure of the infinite general linear group over R. (This work may be thought of as the "K_1-analogue" of the stable theory of projective modules reported in §2.) Our survey concludes with a discussion (§§8-9) of Bass's work on the stable range, with a special emphasis on the case of stable range one which in turn has deep significance on the arithmetic of rings, and on questions concerning the cancellation and substitution of modules with respect to direct sum decompositions.

Throughout this exposition, our aim is not only to survey Bass's work in ring theory and projective modules, but also to point out how much this work has

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[1] Throughout this paper, "f.g." will be used as an abbreviation for "finitely generated".

influenced and shaped the thinking of subsequent researchers in the area. It will be seen that, in a number of new lines of investigation in algebra in the last few decades, it was the decisive pioneering steps of Bass that broke open the new path. Although Bass's work in modules and rings spanned only the decade of the 60s, its impact has been enormous indeed, and will certainly continue to be felt as ring theory moves into the next century.

Writing this article is for me a very pleasant task, though undertaking such a task inevitably involves somewhat of a nostalgic trip down the memory lane. Since it is perhaps not out of place in these Proceedings to talk about one's connections to Hyman, I will indulge myself in a few personal reminiscences below.

As a beginning graduate student at Columbia in the mid-60s, I was more than a little awed by the power and fame of the senior faculty: Professors Lipman Bers, Samuel Eilenberg, Ellis Kolchin, Masatake Kuranishi, Serge Lang, Edgar Lorch, and Paul A. Smith, among others, from whom I took my first and second year graduate courses.[2] After a few skirmishes with functional analysis (taught to me by Professor Lorch and Visiting Professor B. Sz. Nagy), I fell under the spell of Sammy Eilenberg, and positioned myself to become a student of his, hoping to study with him category theory and homological algebra. Under Sammy's guidance, I wrote my first paper [La$_1$], which he kindly communicated to the Proceedings of the National Academy of Sciences (of which he was a member). But then for the year 1966-67, Professor Eilenberg had his Sabbatical coming, and he was to go off to Paris to spend the entire academic year. To go with him to Paris would have been the "cool" thing to do for a mid-career graduate student except, alas, I found myself both financially and linguistically poorly equipped to make the trip. So Professor Eilenberg said to me: "Why don't you stay here, and study with Bass while I am gone?" This was how I became a student of Bass! Hyman was then a young Assistant Professor, who, just a few years ago, was brought from Chicago to Columbia by Sammy himself.

Hyman became an Associate Professor around that time, and in another year he was moved up to Full Professor. Even I could see he must be working on something hot! So I xeroxed all of his papers,[3] and started poring over them assiduously. Hyman's K-theory paper [B$_9$] had just come out in the "Blue Journal"; this and his earlier papers in ring theory and projective modules eventually became a staple of my graduate education. The most challenging open problem in homological algebra in those days was Serre's Conjecture (*ibid.*); many a graduate student in algebra from that era had no doubt tried his/her hand at it — and I was no exception. Hyman, who had generalized Seshadri's solution of the Conjecture in two variables, was the natural leader for this small circle of aspiring graduate students. I still

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[2]The students were nothing short of stellar either! My graduate peers included Mike Engber, Audun Holme, Fred Gardiner, and Irwin Kra; Winfried Scharlau was a visiting student from Germany, and Alexander Mikhalev was an exchange scholar from U.S.S.R.; Bill Haboush, Tony Bak, Mike Stein, Allen Altman, Bruce Bennett, Spencer Bloch, Bob and Jane Gilman were a couple of years behind; Julius Shaneson, Sylvain Cappell and Ethan Akin were "hot-shot" Columbia undergraduates taking graduate courses with us. Each person in this list is now a Professor of Mathematics. Many of them have been chairs, provosts, and deans.

[3]Yes, there were already xerox machines in the mid-60s, though stencils and mimeographed copies were still not entirely out of fashion.

recall that, one time, one of us thought he had a brilliant idea to solve Serre's Conjecture. A few of us excitedly met with Hyman in an impromptu seminar to go over the "idea"; but of course Hyman quickly found the hole. Although we never got anywhere with our fledgling efforts, my fascination with Serre's Conjecture continued, and culminated in the writing of my 1978 Springer Lecture Notes [La$_2$], two years after the Conjecture was fully solved independently by Suslin and Quillen. Hyman's great influence was evident from cover to cover of my modest book.

I spent my last graduate year at Columbia in 1966-67. In that year, Hyman taught what was most probably the first graduate course ever given in algebraic K-theory in the US. Pavaman Murthy had come from the Tata Institute to do post-doctoral work with Hyman, and was in the audience. (In his modest Morningside Heights apartment, Pavaman served what he claimed to be the best coffee in Manhattan. It was free, so I had no reason to disagree; we became good friends.) Some of the lecture notes taken by Murthy, Charles Small and me eventually evolved into Hyman's famed tome [Ba$_{10}$]. At that time, Hyman was busy at work with Milnor and Serre on the Congruence Subgroup Problem for the special linear groups and the symplectic groups. I got lucky and proved several little things about Mennicke symbols and SK_1 of abelian group rings, which earned me a few attributions in [BMS]. Needless to say, I was proud to be mentioned in a paper by such distinguished authors. I got lucky in some other fronts too, and in May, 1967 completed a thesis in algebraic K-theory under Hyman dealing with Artin's Induction Theorem and induction techniques for Grothendieck groups and Whitehead groups of finite groups. In those days, algebraic K-theory meant only K_0 and K_1; even Milnor's K_2 had not been defined yet.

As it happened, I was Hyman's first Ph.D. student. I have always viewed this as a special honor, and I am sure that this fact has helped me a great deal professionally. Now the list of Ph.D. students of Hyman is 25(?) strong. The longevity of Hyman as a thesis advisor is rather strikingly illustrated by the fact that at least several of my mathematical brothers and sisters in this list were not even born yet when I completed my Ph.D. degree at Columbia.

In supervising my work, Hyman never tried to tell me what I should work on. Rather, he let me find my own path, and instilled in me the needed confidence to grow into a research mathematician. What I learned from Hyman was not just mathematics, but how to do mathematics, and, what is perhaps even more important, how to conduct myself as a mathematician. All of this he taught me in the best way — by his own example. For this, and for the many other favors he has rendered over the years, I shall always be grateful.

Part I: Projective (and Torsionfree) Modules

§1. Big Projectives

While the projective modules occurring in number theory, representation theory and algebraic geometry are mostly f.g. ones, non-f.g. projective modules do arise naturally over various kinds of rings, for instance, rings of continuous functions. Thus, the quest for information about non-f.g. projective modules is not a frivolous one.

The first significant result in the study of general (that is, not necessarily f.g.) projective modules was found by Kaplansky in 1958. In his seminal paper $[\mathrm{K}_2]$, Kaplansky proved that, for any ring R, any projective R-module is always a direct sum of countably generated (projective) modules. This result has two interesting consequences. First, any indecomposable projective module over any ring is countably generated; and second, if any countably generated projective module over some ring R is free, then *any* projective R-module is also free. The case of a local ring R provides a particularly striking illustration for the power of the second statement: in this case, Kaplansky used an ingenious argument (reminiscent of the proof of Nakayama's Lemma) to show that any countably generated projective R-module is free, from which it then follows that any projective R-module is free.

Inspired by Kaplansky's result, Bass took up the study of "big" projective modules in $[\mathrm{B}_5]$. The overall theme of $[\mathrm{B}_5]$ is that, under certain fairly mild conditions on a ring R, "big" projective R-modules are necessarily free. To formulate this precisely, Bass introduced the notion of a *uniformly $\aleph$-big* module: for an infinite cardinal $\aleph$, an R-module P is uniformly $\aleph$-big if P can be generated by $\aleph$ elements, and for any ideal $I \subsetneq R$, the R/I-module P/IP cannot be generated by fewer than $\aleph$ elements. For instance, the free module of rank $\aleph$ over a nonzero ring R is uniformly $\aleph$-big. Bass's first result on big projectives is the following converse of this statement:

(1.1) Theorem. *If $R/\mathrm{rad}(R)$ is left noetherian, then, for any infinite cardinal $\aleph$, any uniformly $\aleph$-big projective left R-module is free.*

Bass's proof of this result is modeled upon Kaplansky's proof of his main theorem in $[\mathrm{K}_2]$: one makes a reduction to the crucial case when $\aleph$ is the countable infinite cardinal $\aleph_0$, and in this case one achieves the desired goal by a clever juggling with infinite matrices.

For concrete applications of (1.1), one needs to find situations where we can say, for instance, that all non-f.g. projective modules are uniformly big for some infinite cardinal. Bass showed that this is the case when, say, R is a commutative noetherian ring with only trivial idempotents. Thus, one has the following

(1.2) Corollary. *If R is a commutative noetherian ring with only trivial idempotents, then any non-f.g. projective R-module is free.*

For instance, if $R = k[x_1, \ldots, x_n]$ where k is a field, Serre asked in 1955 (see §2) if every f.g. projective R-module P is free. The result above would give the freeness of P if P was *not* f.g.

The general results of Bass (such as (1.1) and (1.2)) describing the behavior of big projectives have remained essentially unsurpassed to this date. In $[\mathrm{B}_5]$, Bass remarked that these results seemed to indicate that big projective modules "invite little interest." We can say, today, that this is perhaps not quite true. In a recent paper [LR], Levy and Robson have determined the structure of all infinitely generated projective modules over (noncommutative) hereditary noetherian prime rings. Their results showed that, over such rings, there may exist non-f.g. projective modules which are not free and which have rather interesting structures.

Two other results of Bass on projective modules over any ring have become folklore in the subject, so it is fitting to end this section by recalling them. The first of these says that:

(1.3) Theorem. *Any nonzero projective module P over any ring R has a maximal submodule. More precisely, $\mathrm{rad}(R)\,P = \mathrm{rad}(P) \neq P$, where $\mathrm{rad}(P)$ denotes the intersection of all maximal submodules of P.*

This result first appeared in $[B_1\colon \mathrm{p.}\,474]$, and has been used by many authors since. The second result, also involving the Jacobson radical of a ring R, states the following:

(1.4) Theorem. *Let I be any ideal of a ring R such that $I \subseteq \mathrm{rad}(R)$, and P, Q be f.g. projective left R-modules. Then*

$$(1.5) \qquad P \cong Q \ \text{as} \ R\text{-modules} \iff P/IP \cong Q/IQ \ \text{as} \ R/I\text{-modules.}$$

In particular, P is free as an R-module iff P/IP is free as an R/I-module.

This result, which made its first appearance in the literature as Lemma 2.4 in $[B_2]$, is now in virtually every textbook which treats the subject of projective modules, and is well-known to any student knowledgeable about the subject of homological algebra. We should point out that, with Bass's notion of left T-nilpotency on (1-sided) ideals (introduced later in §6), (1.4) can also be given the following "transfinite" formulation, for *any* pair of projective modules P, Q:

(1.6) Theorem. *If I is any left T-nilpotent left ideal in a ring R, then (1.5) holds for any (not necessarily f.g.) projective left R-modules P, Q.*

For a proof of this, see $[La_3\colon (23.17)]$.

§2.   Stable Structure of Projective Modules

In his epoch-making papers $[S_1, S_2]$, Serre established an analogy between projective modules (in algebra) and vector bundles (in topology). This important analogy, which was further promulgated by the work of Swan $[Sw_1]$, enables one to establish the rudiments of a dictionary to translate the language of projective modules into that of vector bundles. Now by the late 1950s, the topology of vector bundles was already rather well-developed; the Serre-Swan analogy mentioned above made it possible, therefore, for algebraists to "predict" (if not prove) theorems in the theory of projective modules based on their knowledge of topological results in the theory of vector bundles.

One of the best known elementary facts about vector bundles is the following. If $[X, Y]$ denotes the set of homotopy classes of maps from X to Y, and $\mathrm{BO}(r)$ denotes the classifying space of the orthogonal group $\mathrm{O}(r)$, then

(2.1) Theorem. *For any connected finite CW complex X of dimension d, the natural map*

$$i_r : [X, \mathrm{BO}(r)] \longrightarrow [X, \mathrm{BO}(r+1)]$$

is surjective for $r \geq d$, and injective (and hence bijective) for $r \geq d+1$.

Now by the classification theorem of vector bundles, $[X, \mathrm{BO}(r)]$ represents the set of equivalence classes of real r-plane bundles over X, and, with this interpretation, the map i_r in (2.1) above is given by adding a trivial line bundle. To make the transfer into algebra, we replace X by a commutative noetherian ring R with only trivial idempotents (so that the Zariski prime spectrum $\mathrm{Spec}(R)$ is a connected space), and replace $[X, \mathrm{BO}(r)]$ by the set $\mathcal{P}_r(R)$ of isomorphism classes of f.g. projective R-modules of rank r. The i_r in (2.1) is then to be replaced by the map given by "adding a copy of R". With such a transfer in place, Serre broke new ground by coming up with commutative algebra techniques to prove the following remarkable analogue of the surjectivity part of (2.1).

(2.2) Theorem. ([S_2: Théorème 1]) *In the above setting, assume that the maximal ideal spectrum* $\max(R)$ *(with the Zariski topology) is a space of dimension*[4] d. *Then the map*

$$(2.3) \qquad\qquad i_r : \mathcal{P}_r(R) \to \mathcal{P}_{r+1}(R)$$

is surjective if $r \geq d$. *In other words, any f.g. projective* R-module of rank $p > d$ *is of the form* $R^{p-d} \oplus P_0$ *for some (projective) module* P_0.

In this setting (and under the same hypotheses), the question whether the map i_r in (2.3) is *injective* for $r \geq d+1$ (in analogy to the second part of (2.1)) begs to be asked. This amounts to the following cancellation question for f.g. projective R-modules P, Q and M: *if* $M \oplus P \cong M \oplus Q$ *and* P *has rank* $> d$, *does it follow that* $P \cong Q$? It would be difficult to imagine that Serre was not aware of this very natural question in 1957-58, but anyhow, Serre did not pursue it in [Se_1].

In his two papers [B_2] and [B_9] (see also the research announcement [BS] written jointly with S. Schanuel), Bass not only answered this question in the affirmative, but also relaxed some of the assumptions in (2.2), and extended all results to a *noncommutative* setting. Working now with a commutative ring A and a module-finite A-algebra R, Bass considered R-modules P which need not be projective or f.g. He defined such a module to be of f-rank $\geq r$ if, at every maximal ideal $\mathfrak{m} \in \max(A)$, the $R_{\mathfrak{m}}$-module $P_{\mathfrak{m}}$ contains an $R_{\mathfrak{m}}$-free direct summand of rank r.

(2.4) Theorem. *In the above (noncommutative) setting, assume that* $\max(A)$ *is a noetherian space of dimension* d.
(1) *(Splitting) If an* R-module P *is a direct summand of a direct sum of finitely presented* R-modules and has f-rank $> d$, *then* P *has a direct summand isomorphic to* R.
(2) *(Cancellation) Let* P, Q *be* R-modules such that P *has a projective direct summand of* f-rank $> d$. *Then, for any f.g. projective* R-module M,

$$(2.5) \qquad\qquad M \oplus P \cong M \oplus Q \Longrightarrow P \cong Q.$$

If we now define $\mathcal{P}_r(R)$ to be the set of isomorphism classes of f.g. projective R-modules of f-rank r, clearly (1) and (2) above imply that:

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[4]The *dimension* of a topological space X is defined to be the supremum of the codimensions of its nonempty closed sets. Here, the codimension of an irreducible closed set F is the supremum of the lengths of finite chains of irreducible closed sets "above" F, and the codimension of an arbitrary closed set C is the infimum of the codimensions of all irreducible closed sets $F \subseteq C$.

(2.6) Corollary. *The map $i_r : \mathcal{P}_r(R) \to \mathcal{P}_{r+1}(R)$ (defined by adding R) is surjective for $r \geq d$, and injective (and hence bijective) for $r \geq d + 1$.*

Just as (2.1) is quantitatively the best result for bundles in topology, both (2.2) and (2.5), (2.6) are quantitatively the best for f.g. projective modules in algebra. We'll mention the usual examples to substantiate this statement in the commutative and noncommutative cases below.

In the commutative case, let R be the real coordinate ring of the sphere S^2, so $R = \mathbb{R}[x, y, z]$, with the relation $x^2 + y^2 + z^2 = 1$. This is a noetherian domain of (Krull) dimension 2. Let $P = \ker(\varphi)$ where φ is the R-epimorphism $R^3 \longrightarrow R$ given by $\varphi(e_1) = x$, $\varphi(e_2) = y$, and $\varphi(e_3) = z$. We have $R^3 \cong R \oplus P$, so P is a f.g. projective R-module of rank 2. Here, P corresponds to the tangent bundle of S^2. Since the tangent bundle is known to be indecomposable, P is also indecomposable (and in particular $P \not\cong R^2$), so the class of P in $\mathcal{P}_2(R)$ is not in the image of the map i_1 in (2.3). This shows that the condition $r \geq d$ for the *surjective* part in (2.6) is the best possible. On the other hand, the fact that $R \oplus P \cong R \oplus R^2$ and $P \not\cong R^2$ shows that the condition $r \geq d + 1$ for the *injective* part in (2.6) is also the best possible.

In the noncommutative setting, a very simple case for application is that of a group ring $R = \mathbb{Z}\pi$, where π is a finite group. Here we take A to be $\mathbb{Z}$ in (2.4), and set $d = 1$. It follows from (2.4)(1) that any f.g. projective R-module is a direct sum of rank 1 projective modules. But, according to a famous example of Swan [Sw$_2$], if π is the generalized quaternion group of order 32, there exists a nonfree rank 1 f.g. projective R-module P such that $R \oplus P \cong R \oplus R$. Since $P \not\cong R$, we see again that, in the conclusion in (2.4)(2), the condition rank$(P) > d$ cannot be further weakened. In Swan's construction, the module P was, in fact, chosen such that, for a suitable maximal order S containing $R = \mathbb{Z}\pi$, $S \otimes_R P \not\cong S$. Therefore, by tensoring up to S, we obtain a similar example (of non-cancellation) over a left and right hereditary module-finite algebra over the one-dimensional ring $\mathbb{Z}$.

Of course, the bounds $r \geq d$ and $r \geq d + 1$ are just *general* bounds for surjectivity and injectivity to hold, respectively, in (2.6). For specific rings, there can be stronger results. For instance, in contrast with Swan's example mentioned above, if π_0 is the (ordinary) quaternion group of order 8, then full cancellation holds for f.g. projective modules over the group ring $\mathbb{Z}\pi_0$, according to a result of J. Martinet [Ma].

Bass's fundamental results (2.4) and (2.6) in the early 60s provided the general framework for much of the subsequent investigations on the Splitting and Cancellation Problems for projective (and more general) modules over various classes of rings. Without giving any details, let us just mention, in this direction, the work of Chase, Mohan Kumar, Murthy, Nori, Sridharan, Suslin, Swan, Towber, Wiegand, and others on the splitting and cancellation of f.g. projective modules over affine algebras. A survey of some aspects of this appears in Murthy's article [Mu] in this volume. On the noncommutative side, there are the important cancellation results of Jacobinski, Guralnick, Levy, Roĭter, Swan and many others for lattices over group rings and orders in separable algebras.

Now it may be said that a cancellation result such as (2.4)(2) is not quite truly in the noncommutative spirit. In this theorem, the ring R in question is a module-finite algebra over a commutative ring A; orders in finite-dimensional separable algebras are also of the same nature. Such rings are simply *not* sufficiently representative of a general noncommutative ring. Later developments show that there are indeed some "truly noncommutative" cancellation theorems, where the cancellation of a module M in (2.5) depends on the "stable range" of the endomorphism ring of M and on the structure of the module P. Such results, which in essence generalize (2.4)(2), were first found by Warfield [Wa$_2$] in 1980 (and in part by Evans in 1973). We shall return to formulate these more general cancellation results after we introduce the notion of stable range of (noncommutative) rings in §8.

§3. Work Related to Serre's Conjecture

In the Serre-Swan analogy between vector bundles and projective modules, the counterpart of the affine n-space k^n over a field k is the polynomial ring in n variables $R = k[x_1, \dots, x_n]$, so vector bundles over k^n correspond to f.g. projective modules over R. From the viewpoint of topology, the (real) affine n-space is contractible, so the vector bundles over it are all trivial. This led Serre to ask the question, in his famous FAC paper [S$_1$], *whether every f.g. projective module over the polynomial ring $R = k[x_1, \dots, x_n]$ is free* (for any field k). An affirmative answer to this question seemed so plausible and convincing to the mathematical public that, almost from the very beginning, it became known under the misnomer of "Serre's Conjecture". This "conjecture" is clearly true when $n = 1$, since in this case $R = k[x_1]$ is a PID, and f.g. projective modules over a PID are well known to be free. In the general case, it is known (essentially from Hilbert's Syzygy Theorem) that any f.g. projective module P over $R = k[x_1, \dots, x_n]$ is stably free, so Serre's Conjecture boils down to a cancellation statement: that $P \oplus R$ free should imply P free.[5] By Bass's Theorem (2.6)(2), this would follow, for instance, if the rank of P is at least $n + 1$.

In the 1960s, Serre's Conjecture became one of the premier open problems in algebra. The fact that the Conjecture was prompted by a natural analogy with vector bundle theory gave it a certain sense of inevitability; on the other hand, the fact that the conjecture can be stated so directly and in such completely elementary terms made it enticing to all. Many algebraists in the 1960s, junior and senior alike, must have tried their hands at solving this famous conjecture. Bass's interest in the structure of projective modules, evident from his first two papers [B$_1$, B$_2$], naturally steered him in this direction.

In 1958, Seshadri [Se$_1$] confirmed Serre's Conjecture in the case of two variables, proving, as he put it, the "triviality of vector bundles over the affine space K^2". As a matter of fact, Seshadri showed more generally that any f.g. projective module P over $A[t]$ "comes from" A if A is a PID [Se$_1$], or the coordinate ring of a nonsingular affine curve over an algebraically closed field [Se$_2$]. These results led Bass to consider the same problem over $R = A[t]$ when A is a Dedekind domain. The desired goal

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[5]In this form, the Conjecture is capable of a completely elementary statement meaningful to any student with a high school background in algebra: *whenever* $f_1 g_1 + \dots + f_r g_r = 1 \in R$, *there is an* $r \times r$ *matrix of determinant* 1 *over* R *with first row* $(f_1, \dots, f_r)$. See [La$_2$: (I.4.5)] for details.

in this case would be the same as Seshadri's — to prove that any f.g. projective P over R "comes from" A, that is, $P \cong R \otimes_A P_0$ for some (necessarily f.g. projective) module P_0 over A.

In the early 60s, Bass succeeded in extending Seshadri's argument, and proved the following result in $[\text{B}_4: (2.4)]$.

(3.1) Theorem. *Let A be a Dedekind ring, $R = A[t]$, and P be a f.g. R-module such that, for any prime ideal $\mathfrak{p} \subset A$, $P/\mathfrak{p}P$ is a torsionfree $(A/\mathfrak{p})[t]$-module. Then P is extended from a f.g. projective A-module; in particular, P is projective.*

This implies, in particular, that any f.g. projective R-module is extended from A. As it turned out, the same result was obtained independently by Serre at about the same time; see $[\text{S}_3]$. In retrospect, this result is perhaps most appropriately called the Seshadri-Bass-Serre Theorem.

In 1964, Bass took up what may be considered a noncommutative version of Serre's Problem. Perhaps not too surprisingly, the noncommutative case turned out to be more tractable. Generalizing the work of P. M. Cohn, Bass succeeded in proving that, if π is a free group (resp. a free monoid), then for any principal ideal domain A, any f.g. projective module over the group ring (resp. monoid ring) $A\pi$ is free. This result appeared in Bass's paper $[\text{B}_7]$, in the first volume of the Journal of Algebra. In the case when π is a free monoid on one generator, it retrieves, of course, Seshadri's theorem in $[\text{Se}_1]$.

Although Bass did not publish further results on Serre's Conjecture after the 1960s, his keen interest in it continued well into the 70s. When Suslin and Vaserstein began to make significant progress on the conjecture in the early 70s, the former Soviet Union was still quite isolated mathematically from Europe and from the U.S. In order to make their latest findings known to the West, Suslin and Vaserstein could only communicate them by letter to Bass. I still remember vividly the AMS Annual Meeting in San Francisco in 1974, in which Bass gave an "impromptu" lecture on the most recent Suslin-Vaserstein results on Serre's Conjecture — to a room-full of people eager to find out how close Serre's Conjecture had come to being solved. One of these "From Russia, with Love" results proclaimed the freeness of projective $k[x_1, \ldots, x_n]$-modules "of rank $\geq 1 + n/2$". This surely looked wonderful, but a high-school algebra question unwittingly came up: when Suslin wrote "$1 + n/2$" in his letter to Bass, did he mean $1 + \frac{n}{2}$, or could he have meant the more ambitious $(1 + n)/2$? It was anybody's guess .... (As it turned out, Suslin did mean $1 + \frac{n}{2}$, as he, perhaps, should.) Later in June that year, in Paris, Bass was to give a similar lecture on the status of Serre's Conjecture in the "Séminaire Bourbaki". The write-up of these survey lectures subsequently appeared in Bass's article $[\text{B}_{12}]$, with the charming title "Libération des modules projectifs ...".

Serre's Conjecture stood open for over twenty years, and was finally proved in 1976, completely independently and almost simultaneously, by D. Quillen [Qu] and A. Suslin [Su]. (For a detailed exposition on this, see $[\text{La}_2]$.) As is often the case in mathematics, however, the solution of one important conjecture was only to be followed by the formulation of a new, more powerful, conjecture. After 1976, Serre's

Conjecture was generalized into the so-called *Bass-Quillen Conjecture,* which states the following:

(3.2)$_d$ *If A is a commutative regular ring[6] of Krull dimension $\leq d < \infty$, then any f.g. projective $A[x_1, \ldots, x_n]$-module is extended from A.*

When A is a field, of course, this gives back the original Serre Conjecture (now the Quillen-Suslin Theorem). When A is a Dedekind ring and $n = 1$, $(3.2)_1$ is the Theorem of Seshadri, Bass, and Serre. By a powerful general technique known as "Quillen Induction", which is gleaned from Quillen's solution of Serre's Conjecture (see [La$_2$: p. 139]), one can reduce the proof of $(3.2)_d$ to a demonstration of the following special case of it:

(3.3)$_d$ *If A is a regular local ring of Krull dimension $\leq d$, then any f.g. projective $A[x_1]$-module is free.*

The best result on $(3.2)_d$ and $(3.3)_d$ known to me is that they are both true when $d \leq 2$, or when A is a formal power series ring over a field; see [La$_2$: p. 138]. On several occasions, I have heard research announcements claiming the general truth of $(3.2)_d$ (for all n and all d), but so far I have not seen any published proofs. Thus, it appears that "Serre's Conjecture" first raised in the 1950s is still very much alive today: it has simply undergone a mathematical metamorphosis and has now become the even more challenging "Bass-Quillen Conjecture". Given this, we can say with a reasonable amount of certainly that the work of Bass on Serre's Conjecture and its generalizations will continue to have its impact on the mathematics of the next century.

§4. Rings with Binary Generated Ideals: Bass Rings

One way to try to prove Serre's Conjecture over $R = k[x_1, \ldots, x_n]$ (k a field) would be to show that any f.g. projective R-module is isomorphic to a direct sum of ideals in R (and then use the fact that $\mathrm{Pic}(R) = \{1\}$ for the unique factorization domain R). The conclusion of Bass's Theorem (3.1) has a rather similar flavor: for any Dedekind ring A, this result says that any f.g. module P over $R = A[t]$ satisfying the torsionfree hypothesis in that theorem is extended from a f.g. projective A-module P_0. Over the Dedekind ring A, P_0 is isomorphic to a direct sum of ideals, so P is likewise isomorphic to a direct sum of ideals in R. Considerations such as this led Bass to the following general question on the decomposition of torsionfree modules:

(4.1) *When is it true that any f.g. torsionfree module over a (commutative) noetherian domain R is isomorphic to a direct sum of ideals?*

Equivalently, when is it true that any f.g. indecomposable torsionfree R-module has rank 1 (that is, isomorphic to an ideal of R)?

As it turned out, this interesting question led Bass to a fruitful program of research. In his paper [B$_4$], Bass not only proved the result (3.1), but also obtained a criterion for the decomposability of all f.g. torsionfree modules into rank one

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[6] A commutative noetherian ring A is said to be regular if $A_{\mathfrak{p}}$ is a regular local ring for any prime ideal $\mathfrak{p} \subset A$.

modules over a noetherian domain R, under a mild assumption on the integral closure $\tilde{R}$ of R. The main theorem (1.7) in [B$_4$] gives the following definitive result.

(4.2) Theorem. *For any commutative noetherian domain R such that $\tilde{R}$ is f.g. as an R-module, the following two conditions are equivalent:*

(1) *Any f.g. torsionfree R-module is isomorphic to a direct sum of ideals;*
(2) *Any ideal in R can be generated by two elements.*

In Bass's proof of this theorem, the hypothesis on $\tilde{R}$ is needed only for the implication $(2) \Rightarrow (1)$.[7] This hypothesis is, of course, a very natural one from the viewpoint of algebraic geometry. As a matter of fact, Bass's proof (in [B$_4$]) for $(2) \Rightarrow (1)$ in the above theorem is based on a rather subtle induction on the length of the R-module $\tilde{R}/R$. The fact that this module has finite length is a consequence of the first conclusion in the following result describing some of the key properties of commutative domains with binary generated ideals.

(4.3) Theorem. *If a commutative domain R has the property that any ideal in R can be generated by two elements, then:*

(1) *R has Krull dimension ≤ 1;*
(2) *every ideal in R is reflexive[8] as an R-module.*

Here, the first conclusion, (1), goes back to I. S. Cohen. In fact, Cohen has proved already in 1949 that the conclusion K-dim $R \leq 1$ will follow if every ideal of the domain R can be generated by k elements for a fixed integer k [C: p. 37, Cor. 1]. Once we have K-dim $R \leq 1$, the conclusion (2) follows from Lemma (1.6) of [B$_4$].

Of course, the driving force behind all of these conditions in (4.2) and (4.3) is the basic example of a Dedekind ring R. For such a ring, the properties (1), (2) in (4.2) and (4.3) are well-known to every student of abstract algebra. The case of Dedekind rings R is precisely the *integrally closed* case of these results, that is, when $\tilde{R} = R$. This is, in fact, the beginning case for Bass's proof for $(2) \Longrightarrow (1)$ in (4.2) by induction on $\text{length}_R(\tilde{R}/R)$. How about the case of non-Dedekind rings? It behooves us to recall the two basic examples mentioned by Bass in [B$_4$: p. 324]:

- Let R be the ring $\{a + 2b\,i : a, b \in \mathbb{Z}\}$, as a subring of the Dedekind ring of Gaussian integers S. Here, $\tilde{R} = S$, and $\text{length}_R(\tilde{R}/R) = 1$. There are only two isomorphism types of nonzero ideals in R, namely, R and $2R + 2i\,R$, so indeed every R-ideal is binary generated.

- For another typical example, let I be the additive submonoid of the non-negative integers generated by 2 and an odd integer $n \geq 3$, and let R be the subring of $S = \mathbb{R}[[t]]$ consisting of power series $\sum_{i \geq 0} a_i t^i$ with $a_i = 0$ for $i \notin I$. Then $\tilde{R} = S$, $\text{length}_R(\tilde{R}/R) = n - 2$, and again it is easy to check that every ideal of R is binary generated.

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[7]In other words, $(1) \Rightarrow (2)$ holds for any commutative noetherian domain R.
[8]An R-module P is said to be *reflexive* if the natural map from P to its double dual P^{**} is an isomorphism.

So far we have quoted two main results ((4.2) and (4.3)) from [B$_4$], which arose from the consideration of torsionfree modules over commutative noetherian domains, and have the common theme of binary generated ideals in such domains. For Bass, there was another important motivation for these results! In fact, (4.3) is very much a part of Bass's research program studying (not necessarily commutative) noetherian rings of finite injective dimensions over themselves. This is a very important program he started in [B$_3$], and continued in [B$_6$]. Let us now explain the connections.

Classically, among the noetherian rings, those that are (say, left) self-injective are the so-called *quasi-Frobenius* (QF) *rings*. These rings of self-injective dimension zero have been extensively studied in the ring theory literature. In generalization of this, Bass sought characterizations of noetherian rings R with $\mathrm{id}(_RR) < \infty$, where "id" stands for the *injective dimension* (of a module). The case of $\mathrm{id}(_RR) \leq 1$ was successfully characterized by Jans [Ja] and Bass [B$_3$: (3.3)], as follows.

(4.4) Theorem. *For any (left and right) noetherian ring R, the following are equivalent:*

(1) $\mathrm{id}(_RR) \leq 1$;
(2) *every f.g. torsionless right R-module[9] is reflexive;*
(3) *If $_RP$ is f.g. projective and $_RN$ is f.g. torsionless, then any short exact sequence* $0 \to P \to M \to N \to 0$ *splits.*

Here, since R is not a domain any more, we do not have a natural notion of torsionfree modules. In their place, Bass introduced the notion of a *torsionless* module: an R-module P is said to be torsionless if the natural map θ_P from P to its double-dual P^{**} is a monomorphism (that is, for any nonzero $p \in P$, $f(p) \neq 0$ for some $f \in P^*$). If R happens to be a commutative domain, a torsionless R-module P is easily seen to be torsionfree; the converse does not hold in general, but does hold if P is f.g.

(We should note in passing that, in stating the above theorem (4.4) in [B$_3$], Bass had another "equivalent" condition: "Every right ideal in R is reflexive." However, while this condition is implied by those in (4.4), Bass's argument for the converse contained a gap, as was later acknowledged in [B$_6$: p.12]. In (6.2) of [B$_6$], this extra condition is restored in the commutative case, in the form "R is Cohen-Macaulay, and every ideal in R is reflexive.")

Let us mention two well-known classes of rings that satisfy the conditions in (4.4). First, consider any noetherian left hereditary ring R. For $E = E(_RR)$ (the injective hull of $_RR$), we have an exact sequence

$$0 \longrightarrow R \longrightarrow E \longrightarrow E/R \longrightarrow 0.$$

Here, the quotient module E/R must be *injective,* by [La$_5$: (3.22)]. Therefore, we have $\mathrm{id}(_RR) \leq 1$. Another interesting (and important) class of noetherian rings R satisfying $\mathrm{id}(_RR) \leq 1$ is given by the group rings kG, where k is any Dedekind ring, and G is any finite group. These rings are "one-step away" from

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[9]Yes, there is a change of side in this statement! Torsionless modules will be defined in the paragraph following the statement of the Theorem.

the quasi-Frobenius rings, in that, if a is any nonzero element in k, the quotient $kG/(a) \cong (k/(a))[G]$ is a well-known example of a QF ring (see [La$_5$: Exer. (15.14)]).

For a commutative noetherian *local* ring R, the condition $\mathrm{id}(_R R) < \infty$ turns out to be one of several equivalent conditions defining a (local) *Gorenstein ring*, and for such a ring R, we have in fact $\mathrm{id}(R) = K\text{-}\dim R$. This homological characterization of a local Gorenstein ring is close in spirit to the usual homological characterization of regular local rings: recall that, by the theorem of Auslander-Buchsbaum and Serre, regular local rings are exactly those noetherian local rings R for which we have $\mathrm{gl.dim}(R) < \infty$, or equivalently, $\mathrm{gl.dim}(R) = K\text{-}\dim R$.

In the case of $K\text{-}\dim R = 0$, the local Gorenstein rings are precisely the local QF rings. In the case of $K\text{-}\dim R = 1$, the earliest manifestation of local Gorenstein rings was in the form of localizations of plane curves, and more generally, complete intersection curves, as was noted by Apéry, Samuel, Gorenstein himself, and Rosenlicht. We refer the reader to Craig Huneke's article in this volume for a thorough survey on the history of Gorenstein rings. In [Hu], Huneke traced the Gorenstein ring notion from the work of the above-named authors to that of Grothendieck and Serre, who defined local Gorenstein rings in the context of duality theory, via the use of dualizing sheaves. After Serre observed the connection to rings with finite self-injective dimensions, Bass wrote the famous "Ubiquity" paper [B$_6$] in 1963 to put the whole theory of Gorenstein rings on a firm footing. One of the basic things he did in this splendid paper was to give a "global" definition for Gorenstein rings: he called a general commutative noetherian ring R Gorenstein if all localizations of R at prime ideals are local Gorenstein rings. If $K\text{-}\dim R$ happens to be finite, this was shown to be equivalent to $\mathrm{id}(R) < \infty$, and again, in this case, $\mathrm{id}(R) = K\text{-}\dim R$. Various other characterizations for Gorenstein rings (e.g. in terms of primary decompositions of ideals, multiplicities, etc.) are given in the "Fundamental Theorem" in §1 of [B$_6$].

To bring the notion of Gorenstein rings to bear on the question of decompositions of modules considered in (4.2), Bass redid and generalized this result in §7 of [B$_6$]. To develop a theory suitable for applications to, say, integral representation theory, Bass considered now a commutative reduced noetherian ring R of Krull dimension 1, with a total ring of quotients K. It is no longer assumed that R is a domain, but we retain the reasonable assumption that $\tilde{R}$, the integral closure of R in K, is a f.g. R-module. (Of course, $\tilde{R}$ is just a finite direct product of Dedekind rings.) The f.g. torsionless R-modules can be seen to be exactly those f.g. R-modules M for which the natural map $M \longrightarrow K \otimes_R M$ is an *injection*. These R-modules may be called *R-lattices*, in analogy with the terminology used in integral representation theory. With this setting in place, Bass considered the following three conditions on R:

(1) *Every R-ideal is generated by two elements.*

(2) *Any ring between R and $\tilde{R}$ is a Gorenstein ring.*

(3) *Every indecomposable R-lattice is isomorphic to an R-ideal.*

Bass's main result in §7 of [B$_6$] is that (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) (and that (3) fails to imply (1) and (2) "only in a situation that can be analyzed completely".[10] This result constitutes an expansion and simplification of (4.2). The novel feature about this result is the emergence of the "hereditarily Gorenstein" condition (2). In the post-1963 literature, a ring R in the above setting satisfying this condition (2) (or equivalently (1)) has been, quite justifiably, called a *Bass ring.*

Later, various authors have obtained new characterizations for Bass rings. For instance, if R and $\tilde{R}$ are as above, then the Bass ring conditions (1), (2) are further shown to be equivalent to each of the following:

(4) (Greither [Gr]) $\tilde{R}$ *is binary generated as an R-module.*

(5) (Levy-Wiegand [LW]) $\tilde{R}/R$ *is a cyclic R-module.*

(6) (Wiegand [Wi]) *Every faithful R-lattice has a direct summand isomorphic to a faithful ideal.*

Yet other characterizations are obtained by Handelman in [Ha$_2$]. Not to be outdone by Bass's "ubiquity" title, Handelman called his own paper "Propinquity of one-dimensional Gorenstein rings".[11] In the 80s, Bass was one of the communicating editors for the *Journal of Pure and Applied Algebra*; it was perhaps not a coincidence, therefore, that the papers of Handelman, Greither, and Levy-Wiegand all appeared in that same journal!

In classical commutative algebra, there is a well-known Steinitz-Chevalley theory for f.g. modules over Dedekind domains. In studying Bass rings, a natural topic to investigate is therefore the classification problem for R-lattices. This problem has been successfully tackled by Levy and Wiegand. In [LW], they show, in generalization of the Steinitz-Chevalley theory, that *any given R-lattice M over a Bass ring R is determined by its genus,*[12] *together with the class of a faithful ideal* $c\ell\,(M) \subseteq R$ *associated with M.* Levy and Wiegand have also obtained very interesting cancellation theorems for projective lattices over Bass rings, and later, Levy [Le] even extended these results from R-lattices to general f.g. R-modules over a specific class of Bass rings.

Of course, the most classical examples of Bass rings are the Dedekind rings (and their finite direct products). The next class of (non-integrally closed) examples are the quadratic orders, that is, $\mathbb{Z}$-orders in a quadratic number field. Any nonzero ideal in such an order R is isomorphic to $\mathbb{Z}^2$, and is therefore binary generated; therefore, R is a Bass order. For yet another class of examples, consider the integral group ring $R = \mathbb{Z}\,G$ for a finite abelian group G. As we have indicated before,

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[10]In analyzing the situations in which (3) fails to imply (1) and (2), however, Bass seemed to have overlooked certain cases. A more complete analysis when R is local was given later by Nazarova and Roĭter in [NR]; for the general case, see the paper of Haefner and Levy [HL]. The main difference between (3) and (1) lies in the fact that (1) is a local property (as Bass had shown in [Ba$_6$: (7.4)]), while (3) is *not* a local property (see [Gr: §2]).

[11]Propinquity := the state of being near in space or in time. Handelman used this term to refer to the fact that the rings between a Bass ring R and its integral closure $\tilde{R}$ are rather "close" to one another.

[12]Recall that two R-modules M and N are said to be in the same genus if $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset R$.

R is always Gorenstein, but it need not be Bass. Bass's result, in this case, shows that (1), (2), (3) above are in fact *equivalent,* and, by the results of Dade and Heller-Reiner, they amount to the fact that the order of G is square-free (see also [Gr: Th. 8.1]). In other words, $|G|$ being square-free is the necessary and sufficient condition for the integral group ring $\mathbb{Z}G$ to be a Bass ring.

To any perceptive reader of the two papers [B$_3$], [B$_6$], there should be little doubt that one of the objectives Bass had in mind for his theory of Gorenstein rings and the decompositions of torsionfree modules was the potential applications to integral representation theory. Bass's explicit mention (in [B$_6$: §7]) of the $R = \mathbb{Z}G$ example in the paragraph above was a clear indication of his vision in this direction. As it turned out, the task of carrying out the program of applying Gorenstein (and Bass) rings to integral representation theory was to fall on the shoulders of researchers on representation modules in the Russian School.

Of course, in integral representation theory, the rings to be considered are no longer commutative; but in a sense they are sufficiently close to commutative rings. To put ourselves into this new setting, we start with a Dedekind ring R with quotient field K, and consider an R-order Λ in a finite-dimensional separable K-algebra A. In this setting, we shall be exclusively concerned with (say, left) Λ-*lattices*; that is, f.g. left Λ-modules L which *embed* into the A-module $K \otimes_R L$ by the natural map). Following Curtis and Reiner [CR: §37], we say that Λ is a *Gorenstein order* if the left regular module $_\Lambda\Lambda$ has the following "weakly injective" property: for any Λ-lattices M and N, any Λ-exact sequence

$$0 \to \Lambda \to M \to N \to 0$$

splits.[13] We then define *Bass orders* by a hereditary property: Λ is said to be a Bass order if every R-order in A containing Λ is a Gorenstein order.

Comparing these definitions with the traditionally well known ones for hereditary orders and maximal orders, we can easily verify the following hierarchy:

{Maximal Orders} $\subset$ {Hereditary Orders} $\subset$ {Bass Orders} $\subset$ {Gorenstein Orders},

where, as indicated, each inclusion is proper. In parallel to the properties (1), (2) and (3) in the commutative case, one can now consider the following three properties of a given R-order Λ:

(a) *Every left ideal of Λ is generated by two elements.*

(b) *Λ is a Bass order.*

(c) *Every indecomposable Λ-lattice is isomorphic to a left ideal of Λ.*

Shortly after the appearance of [B$_6$], Russian workers in integral representation theory mounted an ambitious program to try to determine the exact relationships between the three properties above in the setting of noncommutative R-orders. The definitive results were obtained around 1966-67. In [Ro], Roĭter showed that (a) $\Rightarrow$ (b), and in [DKR], Drozd, Kirichenko, and Roĭter showed that (b) $\Rightarrow$ (c). These results are the best possible, since in general, (b) does not imply (a), nor

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[13]This is actually just an equivalent way to say that $\mathrm{id}(_\Lambda\Lambda) \leq 1$: one can see this, essentially, by applying (1) $\Leftrightarrow$ (3) in (4.4). It can also be seen that Λ being a Gorenstein order is a left/right symmetric property [CR: (37.8)]; hence the omission of any reference to side.

does (c) imply (b). A detailed exposition on the proofs of the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) can be found in §37 of the book of Curtis and Reiner [CR], on which our present discussion is based.

The results of Roĭter, Drozd and Kirichenko are quite deep, involving rather subtle analysis of the decompositions of Λ-lattices. But it was clearly the paradigm of the results of Bass in the commutative case that had guided the Russians in their work in this phase of integral representation theory. Subsequently, Bass orders were used as a fundamental tool in the work of Drozd-Roĭter and Drozd-Kirichenko in their approaches to the characterization of orders of finite representation type (the corresponding work on group rings over rings of algebraic integers having been completed earlier by H. Jacobinski).

In another direction, we should mention that the idea of Gorenstein rings has also found recent applications in noncommutative algebraic geometry: a notion of (noncommutative) "Auslander-Gorenstein rings" has been introduced and studied by K. Ajitabh, S. P. Smith, and J. J. Zhang (see [ASZ]).

Today, in no small measure due to the influence of Bass's paper, Gorenstein rings have lived up to their "ubiquity" billing, and are widely used in number theory, arithmetic and algebraic geometry, commutative (and noncommutative) algebra, theory of invariants, and combinatorics. Even the work of Andrew Wiles [W] on elliptic curves and modular forms leading to his spectacular proof of Fermat's Last Theorem made use of Gorenstein rings at several crucial points: in the Appendix, in Ch. 2 (§1), and then in Ch. 3 on the estimates for the Selmer group. Recall that it was exactly Wiles's earlier attempts to use Euler systems for obtaining the upper bounds on the Selmer group that had led to the "fatal flaw" in his first proof of FLT announced in Cambridge on June 23, 1993. In 1994, finally realizing that the Euler system approach was irreparable, Wiles returned to his original approach in estimating the Selmer group using ideas from Iwasawa Theory. Basically, in [W: Ch. 3], Wiles needed to show that certain minimal Hecke rings, which are known to be Gorenstein rings, are indeed complete intersections. This was eventually accomplished jointly with Richard Taylor in [TW]. The role played by commutative algebra (and Gorenstein rings in particular) in Wiles's paper was described vividly in the following words of his [W: p. 451]:

> "The turning point in this and indeed in the whole proof came in the Spring of 1991. In searching for a clue from commutative algebra I had been particularly struck some years earlier by a paper of Kunz. I had already needed to verify that the Hecke rings were Gorenstein in order to compute the congruences developed in Chapter 2. ... Kunz's paper suggested the use of an invariant (the η-invariant ...) which I saw could be used to test for isomorphisms between Gorenstein rings. A different invariant (the $\mathfrak{p}/\mathfrak{p}^2$-invariant ...) I had already observed could be used to test for isomorphisms between complete intersections. ... Not long afterwards I realized that, unlikely though it seemed at first, the equality of these invariants was actually a criterion for a Gorenstein ring to be a complete intersection."

For related literature, see also Lenstra's paper [Len] on complete intersections and Gorenstein rings. In this paper, Lenstra sharpened Wiles's criterion (in the Appendix of [W]) for a finite $\mathcal{O}$-free local Gorenstein algebra T over a complete discrete valuation ring $\mathcal{O}$ to be a complete intersection. (Lenstra was able to remove the Gorenstein assumption on T.)

Part II: Ring Theory

§5. Semiperfect Rings as Generalizations of Semiprimary Rings

In Part II of this paper, we come to Bass's work in noncommutative ring theory. As we have mentioned in the Introduction, Bass's maiden work [B$_1$], developed from his Chicago thesis in 1959, is a ring-theoretic paper dealing with the homological generalization of semiprimary rings. This work turned out to be one of the most influential ring theory papers written in that period, as can be partly gauged from the following fact. In L. Small's collected reviews [Sm] of ring theory papers published in *Math. Reviews* in 1940-79, an average paper got at most a few cross citations from other reviews, but Bass's paper [B$_1$] managed to pull as many as 29! Usually, a reviewer would only cite a paper in order to indicate the source of a crucial topic or an important idea; the fact that, in its twenty years of existence, [B$_1$] drew as many 29 cross citations from other reviews was almost without parallel in [Sm].

What makes [B$_1$] a masterpiece was the fact that it wove together many of the themes in ring theory and homological algebra that were being developed at that time. On the ring theory side, these themes include: the Krull-Schmidt Theorem (Azumaya version), chain conditions (suitably restricted), maximal and minimal submodules (existence questions), the Jacobson radical (nilpotency questions and lifting of idempotents), and Nakayama's Lemma (for general, not necessarily f.g., modules). On the side of homological algebra (a pretty new subject in 1959), the themes include: projective, injective, and flat modules, projective covers, "Ext" and "Tor" functors, and all kinds of homological dimensions. Quoting from Bass's comments on "Theorem P" in [B$_1$], "this result provides one of those gratifying instances in which several ostensibly diverse notions were shown to be intimately related." Indeed, it seems clear in retrospect that it was this remarkable bridge-building role played by the various results in [B$_1$] which helped secure it a permanent place in the ring theory literature. Only a few years after the appearance of [B$_1$], the key ingredients of the paper (the theory of perfect and semiperfect rings) were incorporated into a standard textbook [L], Lambek's "Lectures on Rings and Modules", ca. 1966. Today, perfect and semiperfect rings continue to be used extensively in a wide variety of ring-theoretic settings.

In this and the next two sections, we'll give a report on Bass's paper [B$_1$] and its impact on noncommutative ring theory. In order to keep the size of these sections within bounds, however, we shall only survey below the first part of [B$_1$] on perfect and semiperfect rings, and will not try to cover its second part on the finitistic homological dimensions of rings.

For the reader's convenience, we first recall a couple of basic definitions. In noncommutative ring theory, a ring R is said to be *semiprimary* if its Jacobson

radical $\mathrm{rad}(R)$ is nilpotent, and the quotient ring $R/\mathrm{rad}(R)$ is (artinian) semisimple. Rings of this type were well known in classical ring theory, and had been studied in part as a viable generalization of one-sided artinian rings by K. Asano, G. Azumaya and T. Nakayama, among others. With the advent of the new style of algebra in the 1950s, homological properties of such rings also attracted attention, and had been explored, for instance, in some of the papers in the "Nagoya series" (ca. 1955-56) on the homological dimensions of modules and rings. Bass had the genius to recognize that the artinian condition on $R/\mathrm{rad}(R)$ is important in its own right, and later (in [B$_9$: p. 504]) defined a ring R to be *semilocal* if it satisfies this condition. In the commutative case, this condition amounts to the finiteness of the number of maximal ideals in R, so Bass's definition of semilocal rings agrees with the usual one in commutative algebra. For noncommutative rings, however, "semilocal" is no longer equivalent to the finiteness of the number of (one-sided or two-sided) maximal ideals. Ring theorists now know that Bass had chosen the right definition (as well as the right name!) for an important class of rings.

Like semiprimary rings, the semiperfect rings and left (right) perfect rings introduced in [B$_1$] are special cases of semilocal rings. Bass was led to these rings partly by the classical idea of an injective hull due to Eckmann and Schöpf. In general, an *injective hull* of a module M is an injective module I containing M as a *large* (or *essential*) submodule, in the sense that:

$$\forall \text{ submodule } S \subseteq I, \quad S \cap M = 0 \Longrightarrow S = 0.$$

Eckmann and Schöpf showed that (over any ring) an injective hull for M always exists, and is unique up to an isomorphism over M. In [Ei], Eilenberg initiated a notion of a *minimal epimorphism* from a projective module to M, and used this notion to study minimal resolutions, homological dimensions and syzygies. Bass observed that, by slightly changing Eilenberg's definition, one gets a precise dual of the notion of an injective hull: according to Bass, a *projective cover* of a module M is a projective module P with an epimorphism $f : P \to M$ such that $\ker(f)$ is *small* (or *superfluous*) in P in the sense that:

$$\forall \text{ submodule } S \subseteq P, \quad S + \ker(f) = P \Longrightarrow S = P.$$

Such a projective cover for M is easily seen to be unique; the only problem is that it may not exist. For instance, in the category of $\mathbb{Z}$-modules, the only objects with projective covers (in the above sense) are the free abelian groups.

The task of studying the existence of projective covers in module categories was taken up by Bass [B$_1$], who defined a ring R to be left perfect (resp. left semiperfect) if every left R-module (resp. cyclic left R-module) has a projective cover.[14] The main work to be done was that of characterizing such rings in terms of other interesting conditions.

In this section, we shall focus on the semiperfect case. Here, Bass's major result is the following.

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[14]The adjective "perfect" is due to Eilenberg, who used the term "perfect categories" in [Ei] for categories of modules possessing projective covers.

(5.1) Theorem. *For any ring R, the following are equivalent:*

(1) *R is left semiperfect;*

(2) *Every f.g. left R-module has a projective cover;*[15]

(3) *R is semilocal, and idempotents in $R/\mathrm{rad}(R)$ can be lifted to idempotents in R.*

In case R is a commutative ring, the above conditions are also equivalent to:

(4) *R is a finite direct product of commutative local rings.*

Note that $(1) \Leftrightarrow (3)$ shows, in particular, the somewhat surprising left/right symmetry of semiperfect rings.[16] The second condition in (3) was already quite well known in ring theory at that time, and had been studied by Kaplansky, Jacobson, and Zelinsky, among others. For instance, if $\mathrm{rad}(R)$ is a nil ideal, this condition is always satisfied (see [L: p. 72, Prop. 1]). Thus, semiperfect rings include all semiprimary rings (which in turn include all one-sided artinian rings). It is in this sense that semiperfect rings (and the 1-sided perfect rings to be discussed later) are homological generalizations of the classically well known semiprimary rings.

The equivalence $(1) \Leftrightarrow (4)$ in Theorem 5.1 shows that, for commutative rings, the notion of semiperfect rings has essentially nothing new to add to the existing theory. But the interesting case is that of *noncommutative* semiperfect rings. Since a major source of noncommutative rings is the class of endomorphism rings of modules, it is significant to ask when such endomorphism rings are semiperfect. The answer to this question is contained in the following result, which is a remarkable extension of the well-known classical theorem that the endomorphism ring of a module of finite length is always semiprimary.

(5.2) Theorem. *Let M be a right module over a ring S. Then the endomorphism ring $\mathrm{End}_S(M)$ is semiperfect iff M has a finite Azumaya decomposition, that is, a decomposition $M = M_1 \oplus \cdots \oplus M_n$ such that each endomorphism ring $\mathrm{End}_S(M_i)$ is local.*

Although this result was not explicitly stated in [Ba₁], it can be proved easily using the techniques of that paper. (An explicit proof can be found in [La₃: (23.6)].) In some sense, (5.2) shows the "ubiquity" of semiperfect rings. For instance, for any *injective* module M_S, if M has finite uniform dimension, then $\mathrm{End}_S(M)$ is semiperfect (and conversely). As a special case of (5.2), we also see that a ring R is semiperfect iff the regular module R_R has a finite Azumaya decomposition; or, in terms of idempotents, *iff there is a decomposition $1 = e_1 + \cdots + e_n$ where the e_i's are mutually orthogonal idempotents with each $e_i R e_i$ a local ring.* The existence of such a decomposition makes it possible to generalize a considerable amount of the elementary theory of artinian rings to a semiperfect ring R, including, for instance: the classification of f.g. projective R-modules, the construction of projective covers for f.g. R-modules, the definition of a Cartan matrix, block decomposition and

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[15]It is worth noting that, according to later work of F. Sandomierski ([Sa₁], [Mue]; see also [La₄: (24.3)]), this condition (2) is also equivalent to the following: (2)′ *Every simple right R-module has a projective cover.* In a same vein, Sandomierski proved that a ring R is right perfect iff every every *semisimple* right R-module has a projective cover.

[16]In fact, the left-right symmetric characterization of semiperfect rings in (5.1)(3) is often used as their *definition* in the literature: see, for instance, [La₃: p. 346].

basic ring for R, etc. These are parts of the foundational material for the theory of artinian rings, first developed (by Brauer, Osima and others) in the context of group algebras of finite groups for applications to the theory of modular representations. It is gratifying to see that a large part of this well-known classical theory can be carried over *verbatim* to semiperfect rings.

The ultimate justification for the introduction of the class of semiperfect rings lies in the fact that, besides semiprimary rings, there are also many other natural classes of rings which turn out to be semiperfect. Let us make a list of some such classes below.

- If N_S is any module of finite uniform dimension over a ring S and M is its injective hull, then the endomorphism ring $\text{End}(M_R)$ is semiperfect. (This is essentially a rehash of the remark on injective modules we made in the last paragraph.)
- If a semilocal ring R is 1-sided self-injective, then R is semiperfect; see [La$_5$: (13.4)].
- Certain semigroup rings called *Kupisch rings* by H. Sato are semiperfect; see [Sat: (5.1)].
- Any right serial ring (that is, a ring R such that R_R is a direct sum of uniserial modules) is a semiperfect ring; see, e.g., [F$_2$: p. 81].
- It is well-known to researchers in duality theory that if a ring R admits a Morita duality into some other ring S, then R and S must be semiperfect rings. Indeed, if U is an (S, R)-bimodule defining the duality, then there is a duality between the Serre subcategories of U-reflexive right R-modules and left S-modules. Since (left) S-modules have injective hulls, it is not difficult to check from the above duality that f.g. (right) R-modules have projective covers. Thus, R, and hence also S, must be semiperfect rings. This basic observation is due to Barbara Osofsky. As a special case, it follows that any cogenerator ring[17] is a semiperfect ring.
- More generally, Sandomierski [Sa$_2$: p. 335] has shown that right linearly compact rings are semiperfect, and Azumaya [Az$_1$], Osofsky [Os] and others have shown that right PF (pseudo-Frobenius) rings[18] are also semiperfect.

The notion of semiperfect rings has also been generalized in several directions. The following are two of them.

- First, prompted by Bass's characterization (5.1)(2) for a semiperfect ring, ring-theorists have come up with a slightly more general notion: a ring R is said to be F-semiperfect if every *finitely presented* left R-module has a projective cover. This terminology is due to Oberst and Schneider [OS]: "F" here stands for "finite". Clearly, every semiperfect ring is F-semiperfect. In parallel to (5.1)(3), there is the following characterization of an F-semiperfect ring: R *is F-semiperfect iff it is semiregular*; that is, idempotents of $R/\text{rad}(R)$

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[17]A ring R is called a *cogenerator ring* if R is a cogenerator both as a left and as a right R-module. A cogenerator ring is also known as a *Morita ring* in the literature.

[18]A ring R is called a *right* PF *ring* if any faithful right R-module is a generator. For a list of many other characterizations of such rings, see [La$_5$: (19.26)]. The cogenerator rings in Footnote (17) are precisely the 2-sided PF-rings.

can be lifted to R, and the epimorphic image $R/\mathrm{rad}(R)$ of R is a von Neu-
mann regular ring (instead of a semisimple ring). (See, e.g. [OS: (1.2)],
and [Ni$_3$, Ni$_4$].) There is a large supply of such rings. In fact, if M is any
quasi-injective module,[19] then the endomorphism ring $\mathrm{End}_R(M)$ is always
a semiregular ring. This fact, first proved by Faith and Utumi [FU], is one of
the underpinnings for the Findlay-Lambek-Utumi theory of maximal rings
of quotients; a self-contained proof of it can be found in [La$_5$: (13.1)].

- In the theory of direct sum decompositions of modules, there is an important
  class of rings called "exchange rings" that are formally christened by Warfield
  in [Wa$_1$]. (A ring R is said to be an exchange ring if R_R satisfies the exchange
  property introduced in the work of Crawley and Jónsson [CJ]. For more
  details on (and characterizations of) such rings, see [Ni$_4$].) The following
  hierarchy shows exactly how semiregular rings and exchange rings compare
  with semiperfect rings:

  {Semiperfect Rings} $\subset$ {Semiregular Rings} $\subset$ {Exchange Rings},

  where the second inclusion was shown by Warfield ([Wa$_1$: Th. 3]; see also
  [Ni$_4$: Cor. (2.3)]). Here, both inclusions are *proper*, as indicated. It turns
  out that the main difference between semiperfect rings and exchange rings
  lies in a finiteness condition. In fact, by combining Nicholson's results [Ni$_1$:
  (4.3)] and [Ni$_4$: (1.9)], one sees that *semiperfect rings are just the exchange
  rings which do not have infinite sets of nonzero orthogonal idempotents*; this
  is stated as (8.4C) in Faith's recent book [Fa$_3$]. Note that this statement is
  much easier to prove if we replace the word "exchange rings" by "semiregular
  rings", since it is well-known that, whenever idempotents can be lifted from
  $R/\mathrm{rad}(R)$ to R, any countable set of nonzero orthogonal idempotents in
  $R/\mathrm{rad}(R)$ can be lifted to a similar set in R [La$_3$: (21.25)], and that a von
  Neumann regular ring is semisimple if it has no infinite sets of nonzero
  orthogonal idempotents ([Go$_1$: (2.16)], [La$_5$: Exer. 6.29]).

Exchange rings are worthy of study in ring theory since they generalize semiper-
fect rings, and they form a fairly broad class of rings. Some known facts about
semiperfect rings turn out to be true for exchange rings; the proofs of them are
sometimes clearer when they are expressed in the context of exchange rings. For
instance, Müller's well-known result [Mue] that *any projective (right) module over
a semiperfect ring R is isomorphic to a direct sum $\bigoplus_i e_i R$ (where $e_i = e_i^2 \in R$)*
generalizes to, and is quite easy to prove over, any exchange ring R; see [Wa$_1$: Th. 1].
Apply this to a local ring R and you'll retrieve Kaplansky's classical result (men-
tioned in §1) that any projective R-module is free.

Finally, we should mention that the notion of semiperfect rings has also been
successfully extended to a module-theoretic setting: a module M over a ring R is
said to be semiperfect if every quotient of M has a projective cover. (Thus, R is
semiperfect iff the module $_R R$ is.) The theory of semiperfect modules was initiated
in the projective case in Mares [M] and Kasch-Mares [KM], and has been studied
further by Nicholson [Ni$_2$, Ni$_3$], and Azumaya [Az$_2$, Az$_4$, Az$_5$], among others. For
a detailed treatment of semiperfect modules in the general case, see the textbook

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[19]A module M is said to be *quasi-injective* if, for any submodule $N \subseteq M$, any homomorphism
from N to M can be extended to an endomorphism of M. Needless to say, quasi-injective modules
include all injective ones.

of Kasch [Ka]. Ever since [B$_1$] appeared in 1960, semiperfect rings (and modules) have been further studied and extensively utilized in many papers in ring theory. To reflect this trend, I have devoted the last chapter of my ring theory graduate text [La$_3$] to an introductory exposition on the theory of (perfect and) semiperfect rings.

§6. Perfect Rings and Restricted DCC

In this section, we come to Bass's remarkable characterizations of left perfect rings; recall that these are, by definition, rings all of whose left modules have projective covers. Since left perfect rings are obviously semiperfect, we expect that one of the characterizations should be a strengthening of the condition (3) in (5.1); this is given by the condition (2) below. The other conditions will be commented upon later.

(6.1) Theorem. *For any ring R with Jacobson radical $J = \mathrm{rad}(R)$, the following are equivalent:*

(1) *R is left perfect;*

(2) *R is semilocal, and J is left T-nilpotent, that is, for any $a_1, a_2, \ldots \in J$, $a_1 a_2 \cdots a_n = 0$ for some n;*

(3) *R is semilocal, and every nonzero left R-module has a maximal submodule;*

(4) *Every flat left R-module is projective;*

(5) *R satisfies DCC on principal right ideals;*

(6) *Any right R-module satisfies DCC on cyclic submodules;*

(7) *R is semilocal, and every nonzero right R-module has a minimal submodule;*

(7)′ *R has no infinite sets of nonzero orthogonal idempotents, and every nonzero right R-module has a minimal submodule.*

It is worth pointing out that left perfect rings are interesting mostly when they are not semiprime. In fact, as soon as a left perfect ring R is semiprime, then R is semisimple; see, e.g. [La$_3$: (10.24)].

According to Bass, the "T" in "T-nilpotency" in (2) stands for "transfinite". Note that this condition in (2) comes somewhere between J being nilpotent and J being nil.[20] Similar nil-ness conditions have appeared before in the work of Levitzki on the Levitzki radical, but Bass was the first one to realize the role of T-nilpotency in the study of modules and restricted chain conditions on 1-sided ideals. As it turned out, the successful use of the T-nilpotency condition played a pivotal role in the proof of (6.1). Results such as (1.6) showed further the efficacy of the T-nilpotency conditions.

We shall now comment on the other conditions in Theorem (6.1). Along with (1), the condition (4) is of a homological nature; it can be slightly rephrased as follows.

(4)′ *Every left R-module has the same flat as projective dimension.*

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[20]In particular, the characterization (2) implies immediately that semiprimary rings are both right and left perfect. In general, however, right perfect and left perfect are *not* equivalent conditions, as was already pointed out by Bass.

Since, in general, flat modules are precisely direct limits of projective modules (by the theorem of Lazard and Govorov), (4) can also be stated in the following form:

(4)″ *Direct limits of projective left R-modules are projective.*

Bass's proof for $(4)'' \Rightarrow (5)$ is based on a very ingenious analysis of the left R-module with generators $x_1, x_2, \ldots$, and relations $x_n = a_n x_{n+1}$ for all n, where $a_1, a_2, \ldots$ are given elements in R.

Note that the first four conditions in (6.1) ((1)–(4)) are conditions on *left R*-modules, while the last four conditions ((5)–(7)′) are on *right* ones! This switch from left modules to right modules, albeit not new for Bass (see Footnote (9)), is in fact one of the inherent peculiar features of his Theorem (6.1). Unfortunately, because of this unusual switch of sides, Theorem (6.1) is often misquoted in the literature, sometimes even in authoritative sources: see, for instance, [KS: Thm. 2, p. 57]. In a couple of standard textbooks, Bass's left (resp. right) T-nilpotency condition was renamed the "right (resp. left) vanishing condition"; in another textbook, the author simply switched Bass's definitions of left and right T-nilpotency — and did so without even alerting the reader to the difference! All of this evidently further compounded the confusion. We hereby urge all future authors to exercise restraint in changing existing definitions, and to check their statements very carefully when quoting Bass's Theorem (6.1).

While clearly (2) and (5) in (6.1) are purely ring-theoretic conditions, Bass's proof of $(2) \Longrightarrow (5)$ was routed through the homological condition (4) (or its equivalent versions (4)′, (4)″). This raised the challenging question whether it is possible to give a direct proof for $(2) \Longrightarrow (5)$ without using any homological algebra.[21] Such a proof was eventually found by Rentschler in [Re].

Let us now discuss in detail the interesting condition (5), which is a natural weakening of the usual artinian condition on right ideals. Here, Kaplansky's ideas played a key role. In his work with Arens on the topological representations of algebras in the late 40s, Kaplansky [K₁] was motivated to consider the condition stipulating the stabilization of all chains of the form

$$aR \supseteq a^2 R \supseteq a^3 R \supseteq \cdots$$

for all elements a in a ring R. This condition was shown to be left/right symmetric many years later by Dischinger, and defines the class of *strongly π-regular rings*. We will not try to justify the somewhat clumsy terminology here, for this can be done only by making a digression into some other definitions not essential for our purposes. Suffice it to say that, in the case of *commutative* rings R, strong π-regularity amounts to R having Krull dimension 0. And, as examples of non-commutative strongly π-regular rings, Kaplansky mentioned the class of algebraic algebras over a field.

In an appendix to [K₁], Kaplansky pointed out that it would also be natural to consider the DCC for *all* principal right ideals in a ring, that is, the condition (5) in Theorem 6.1. Kaplansky noted that the Russian mathematician Gertschikoff

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[21] The reverse implication $(5) \Longrightarrow (2)$ can indeed be done directly without invoking homological tools; see, for instance, the exposition in [La₃: p. 369].

had considered rings satisfying this condition as far back as 1940, and obtained characterizations of such rings that are without nonzero nilpotent elements.[22] In retrospect, this result of Gertschikoff was certainly a harbinger for the equivalence of the conditions (2) and (5) in Bass's Theorem 6.1. By a rather strange coincidence, 1959-61 turned out to be the years in which the minimum condition (5) for principal right ideals was destined to blossom: during this period, along with the publication of $[B_1]$ came two papers of Faith $[Fa_1]$ and three papers of Szász $[Sz]$ on the same topic. Bass invented the term "left perfect" for his rings (after Eilenberg), while Faith used the English acronym "MP-ring", and Szász introduced the German acronym "MHR-ring." Carl Faith told me that, in his first meeting with Bass in the early 60s, they compared notes on their respective works on rings with DCC on 1-sided ideals, but found surprisingly that what they did with these rings had almost nothing in common!

Going beyond Bass's paper $[B_1]$, we should mention several significant results obtained later by other authors, which were directly inspired by Theorem 6.1. Prior to the publication of $[B_1]$, Chase had proved in his Chicago thesis that, over any semiprimary ring R, DCC holds for f.g. submodules of any (left or right) R-module.[23] Prompted by this, Bass asked if, in any left perfect ring, f.g. right ideals satisfy the DCC. This was affirmed in 1969 by Björk ($[Bj_1]$, see also $[La_4: (23.3)]$) who proved the amazing result that, whenever a module (over *any* ring) satisfies DCC on cyclic submodules, it also satisfies the DCC on f.g. submodules. In particular, this extends Chase's result to left perfect rings, showing that in the condition (6) in Theorem 6.1, the word "cyclic" can be replaced by "f.g.". One year later, Jonah $[Jo]$ added "infinitely many" more equivalent conditions to the list in (6.1):

$(0)_n$ *Any left R-module satisfies* ACC *on its n-generated submodules.*

(Here, n is any natural number.) Remarkably, these new characterization of left perfect rings are in terms of *ascending* chain conditions; also, we are now back full circle to *left* R-modules! Note that, in particular, Jonah's result implies "Rings with the minimum condition for principal right ideals have the maximum condition for principal left ideals." This was, in fact, the title of Jonah's paper $[Jo]$.

To continue our discussion on left perfect rings, we take another look at the list of equivalent conditions in (6.1). Since the condition (7) is equivalent to the ostensibly weaker condition $(7)'$, an obvious question one can ask is whether (3) can likewise be weakened to:

$(3)'$ *R has no infinite sets of nonzero orthogonal idempotents, and every nonzero left R-module has a maximal submodule.*

Note that this condition is in a way "dual" to the condition $(7)'$ (just as (3) is dual to (7)), so it seems tempting to add $(3)'$ to the list of equivalent conditions in (6.1). An open problem proposed in $[B_1]$ is, indeed, whether $(3)'$ is a characterization for left-perfectness of a ring R. Some authors have referred to an affirmative answer to this as "Bass's Conjecture", although in fact Bass merely raised the question (see

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[22]In fact, Gertschikoff's characterization worked more generally for rings possibly without an identity element.

[23]For a self-contained proof of this, see Faith's paper $[Fa_2: p. 189]$.

$[B_1: p. 471]$). The answer to this question is possibly a bit surprising: it is "yes" in the commutative case, and "no" in the general case, as is shown by Koifman [Ko], and partly by Cozzens, Hamsher, and Renault. To give an idea of how these conclusions were obtained, we proceed as follows.

In view of (3) and (3)′, it is of interest to isolate the condition: "any nonzero left R-module has a maximal submodule". Let us say that R is a *left-max ring* if this condition is satisfied. It is not hard to see that any left-max ring has a left T-nilpotent Jacobson radical $[La_4: (24.6)]$. In the commutative case, one has the following characterization of a (left)-max ring, due to Hamsher, Renault and Koifman; see $[La_4: (24.9)]$.

(6.2) Theorem. *A commutative ring R is (left)-max iff* $\mathrm{rad}(R)$ *is T-nilpotent and $R/\mathrm{rad}(R)$ is a von Neumann regular ring.*

It follows, in particular, from this result that a commutative max-ring R is semiregular. If, in addition, R has no infinite sets of nonzero orthogonal idempotents, then $R/\mathrm{rad}(R)$ must be semisimple by an argument we gave in §5. Since $\mathrm{rad}(R)$ is T-nilpotent, R is perfect, which answers Bass's question affirmatively for commutative rings.

To treat Bass's question in the general case, we use the notion of a left V-ring — a ring whose simple left modules are injective. A well-known characterization for such a ring R is that every left R-module has a zero radical ($[La_5: (3.75)]$); in particular, R must be a left-max ring. Using differential algebra, Cozzens [Co] constructed remarkable examples of left V-domains that are not division rings; such domains clearly satisfy (3)′ above, but are *not* left perfect rings, thus answering Bass's question in the negative. Moreover, Cozzens's rings are simple, principal left/right ideal domains. The emergence of rings of this type served as an important turning point for the theory of simple noetherian rings; see, e.g. the book of Cozzens and Faith [CF].

Many other characterizations of left perfect rings are known. Without any attempt at completeness, let us mention a few below.

- P. A. Griffith, B. Zimmermann-Huisgen and others have characterized left perfect rings in terms of conditions similar to (6.1)(4). In the homological theory of modules, an R-module P is said to be *locally projective* if every R-epimorphism $f : Q \longrightarrow P$ is locally split (in the sense that, for every $p \in P$, there exists $g \in \mathrm{Hom}_R(P, Q)$ such that $fg(p) = p$). It is easy to see that such a module P must be flat, so the locally projective modules form a class between the class of projective modules and that of flat modules. In $[Zi_2]$, Zimmermann-Huisgen showed that *a ring R is left perfect iff every locally projective left R-module is projective.* This is, therefore, a variant of the criterion (4) for left perfect rings.

- In Theorem 10 *loc. cit.*, Zimmermann-Huisgen has also characterized left perfect rings in terms of $\aleph_1$-separable modules; these are modules P with the property that any countable subset of P is contained in a countably generated direct summand of P.

- In the work of Harada-Ishii [HI], Yamagata [Ya], Zimmermann-Huisgen and Zimmermann [ZZ$_1$], some characterizations of left perfect rings are given in terms of the exchange property of projective left R-modules.

- It seems to be a folklore result that a ring R is left perfect (resp. semiperfect) iff every left R-module P (resp. f.g. left R-module P) is "supplemented", that is, every submodule of P has an addition complement. (See, e.g. [Wis: (42.6), (43.9)].) Recently, Keskin [Ke] has further extended these characterizations by using the more general notion of $\oplus$-supplemented modules.

- Optimally, characterizations for left perfect rings should be generalizable to characterizations for the endomorphism ring of a given module to be left perfect. Some results in this direction can be found in [AF$_2$: §29], [Wis: (43.10)], and [Az$_3$].

In closing, we should mention the fact that much of Bass's work on left perfect rings can be extended to *functor categories.* In general, if $\mathbf{A}$ is a small additive category, we may view $\mathbf{A}$ as a generalization of a ring (it is a "ring with several objects"), and even more significantly, we may view Add($\mathbf{A}$, $\mathbf{Ab}$) (the category of additive functors from $\mathbf{A}$ into the category of abelian groups) as a generalization of the category of modules over a ring. In view of this, it is not surprising that various module-theoretic notions can be generalized to notions concerning functors in Add($\mathbf{A}$, $\mathbf{Ab}$). As it turns out, after defining flat functors, projective covers of functors, and DCC for f.g. subfunctors, etc., one can completely "transfer" Bass's Theorem (6.1) into a theorem on the functor category Add($\mathbf{A}$, $\mathbf{Ab}$). This leads then to the notion of a *left (resp. right) perfect* additive category $\mathbf{A}$. For a full account of all this (as well as various other relations between perfect rings and model-theoretic algebra), see the book of Jensen and Lenzing [JL].

Now all of this is not just generalization for generalization's sake! For instance, one can apply it back to the case when $\mathbf{A}$ is a certain additive subcategory of a module category over some ring R. By doing so, one is sometimes able to make interesting connections, and even prove nontrivial results. For instance, if $\mathbf{A}$ is the category of all finitely presented modules over a ring R, it turns out that the study of the perfectness of $\mathbf{A}$ leads to various insights about (and characterizations of) the so-called pure semisimple rings. For a detailed formulation of this, see [JL: Thm. B.14]. As a matter of fact, we shall return to the theme of pure semisimple rings in the next section in the context of Auslander's representation theory of artinian rings.

Needless to say, none of the work discussed above would have been possible without the pioneering effort of Bass in [B$_1$].

§7. Perfect Rings and Representation Theory

In mathematics, a good notion or a good theorem often has a way of finding surprising connections to things to which it might have seemed unrelated at first. As it turned out, Bass's notion of perfect rings and his various results on them provide such an example. The unexpected connections are to the representation theory of artinian rings, and the study of the general decomposition theory of modules into direct sums. In this section, we'll give a short account on some of these connections;

our discussion will culminate in an open question in noncommutative ring theory which has remained unanswered to this date. I thank N. V. Dung for suggesting that I include such a discussion here, and for explaining to me the main results and references in this area of research.

To set the stage for our discussion, we first recall the following basic notion in representation theory. A ring R is said to be of *finite representation type* (FRT) if it is left artinian and there are only finitely many (isomorphism types of) f.g. indecomposable left R-modules. According to a result of Eisenbud and Griffith [EG], this notion is left/right symmetric, so we are justified in suppressing the word "left" or "right" in referring to rings of FRT. The study of these rings, in the form of finite-dimensional algebras over fields, goes back a long way to Brauer, Thrall, Kasch, Kneser, Kupisch, and others. It turns out that rings of FRT have a rather subtle relationship to Bass's perfect rings, which we'll now try to explain.

We start with a notion introduced by Anderson and Fuller [AF$_1$] in the decomposition theory of modules. A decomposition of an R-module M into $\bigoplus_{i \in I} M_i$ ($M_i \neq 0$) is said to "complement direct summands" if, for each direct summand N of M, there exists a subset $J \subseteq I$ such that $M = N \oplus \bigoplus_{j \in J} M_j$. (Note that if such a decomposition for M exists, the summands M_i are necessarily indecomposable.) Using this terminology and going beyond [B$_1$], Anderson and Fuller obtained in [AF$_1$] (ca. 1972) the following new characterization of left perfect (and semiperfect) rings:

(7.1) Theorem. *A ring R is left perfect (resp. semiperfect) iff every projective left R-module (resp. f.g. projective left R-module) has a direct sum decomposition that complements direct summands, iff the free left R-module $R \oplus R \oplus \cdots$ (resp. $R \oplus R$) has a direct sum decomposition that complements direct summands.*

In view of this theorem, it is natural to ask when does *every* left R-module admit a decomposition that complements direct summands. It turns out that the answer to this question also involves left perfect rings, albeit in different way. Given a ring R, let $_RU$ be the direct sum $\bigoplus_i U_i$, where the U_i's consist of one isomorphic copy of each finitely presented left R-module. Now let E be the subring (without identity) of $\mathrm{End}_R(U)$ consisting of endomorphisms f such that $U_i f = 0$ for almost all i; E is called the "left functor ring" of R (for reasons that we shall not elaborate on here). In 1976, building upon the results of Auslander and Harada, Fuller [Ful] proved the following remarkable result (see also Simson's paper [Si$_1$]).

(7.2) Theorem. *For any ring R, the following are equivalent:*
(1) Every left R-module has a decomposition that complements direct summands;
(2) Every left R-module is a direct sum of f.g. submodules;
(3) The left functor ring E associated to R (defined above) is left perfect.
According to a theorem of Chase, these conditions (specifically (2)) imply that R is a left artinian ring.

In (3) above, of course, we'll need to use the notion of a left perfect ring without an identity. This is not a big problem; in fact, with essentially the same definition of left perfectness, Harada [H] has shown that much of Theorem 6.1 can be proved for rings without 1 but "with enough idempotents" (see also [Wis: §49]).

The condition (2) in (7.2) expresses a very desirable module-theoretic property (arbitrary left modules can be "constructed" from f.g. ones) that has also been studied by Auslander [Au$_2$], Gruson and Jensen [GJ] (and many others in the commutative case). In [GJ], it was shown that (2) is equivalent to R having "pure left global dimension zero", that is, if every pure short exact sequence of left R-modules splits.[24] In 1977, Simson [Si$_2$] introduced the shorter term "left pure semisimple" for rings R with this property, and later, expanding on the work of Chase and Warfield, Zimmermann-Huisgen [Zi$_2$] showed that the property (2) is also equivalent to:

(4) *Every left R-module is a direct sum of indecomposable submodules.*

Therefore, *any* of (1), (2), (3), (4) is a characterization for the left pure semisimplicity of a ring R.

Now finally, we come to the connection between pure semisimplicity and rings of FRT. This is given by the following theorem of Fuller and Reiten [FR] (ca. 1975), which completed the earlier work of Auslander, Ringel and Tachikawa:

(7.3) Theorem. *A ring R is of FRT iff it is both left and right pure semisimple; that is, iff every left and every right R-module is a direct sum of f.g. submodules.*

The major question that remains in this area of study is whether *left* pure semisimplicity is equivalent to *right* pure semisimplicity. An affirmative answer to this question is known as the "Pure Semisimplicity Conjecture" (PSC) in ring theory. If this Conjecture holds up, then the sufficiency part of the theorem above would say that left pure semisimple rings are of FRT. As a positive evidence for this, we mention the "sparsity" result of Prest [Pr], Zimmermann-Huisgen and Zimmermann [ZZ$_2$], which states that, over a left pure semisimple ring R, there exist only finitely many (isomorphism types of) indecomposable left R-modules *of any given composition length.* This means R comes indeed "reasonably close" to being of finite representation type. Also, it is worth noting that, according to results of Simson [Si$_3$] and Herzog [Her], the general form of PSC would follow as soon as one could prove that a left pure semisimple ring is necessarily *right* artinian. For more perspectives and recent results on PSC, see [Si$_4$] and the literature referenced therein.

For connections with the work of Bass, we note that, with the help of left functor rings, PSC can actually be formulated entirely in terms of the notion of perfect rings. We need the following additional fact, which can be gleaned from the work of Auslander [Au$_1$] and Fuller [Ful] (see also Wisbauer's book [Wis: (54.3)]).

(7.4) Proposition. *A ring R is left and right pure semisimple iff its left functor ring E (defined in the paragraph preceding (7.2)) is both left and right perfect.*

Granted this and (7.2), the issue of whether a left pure semisimple ring R needs to be right pure semisimple boils down to testing whether the perfectness of E on the left would imply its perfectness on the right. In an earlier footnote (Footnote (20)), we have mentioned Bass's observation that, in general, a left perfect ring need

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[24]A short exact sequence of left R-modules is said to be *pure* if it remains exact upon tensoring by any right R-module.

not be right perfect. The above considerations show, however, that PSC reduces to a "left perfect implies right perfect" statement, *for the special class of left functor rings.*

As we have mentioned at the end of §6, Bass's ideas of using projective covers and the condition "flat $\Rightarrow$ projective" for module categories have been successfully carried over to categories of functors. Such a generalization has proved to be quite fruitful for the representation theory of artinian rings; see, *inter alia*, the work of Auslander [Au$_1$] and Simson [Si$_1$]. Bass's left and right T-nilpotency conditions for ideals perhaps also inspired Auslander's definition in [Au$_1$] of noetherian and conoetherian conditions for families of homomorphisms between modules. In fact, for a left artinian ring R, the left and right T-nilpotency conditions on the Jacobson radical of the left functor ring E are precisely equivalent to the noetherian and co-noetherian conditions for families of homomorphisms between f.g. indecomposable left R-modules. From these and the preceding discussions, we see that Bass's early work in [B$_1$] has certainly played a substantial and rather interesting role in the later development of the representation theory of artinian rings.

§8. Stable Range of Rings

The notion of stable range for rings is another great invention of Bass that has proved to be of lasting importance in algebra and ring theory. This notion was originally introduced by Bass (in [B$_9$: p. 14]) for the study of stabilization questions in algebraic K-theory, or at least for working with the functor K_1. The definition of stable range goes as follows.

(8.1) Definition. We say that a positive integer n is *in the stable range* of a ring R (or, more informally, that R "has stable range n") if, whenever $a_1 R + \cdots + a_{m+1} R = R$ for $m \geq n$ (where all $a_i \in R$), there exist elements $x_1, \ldots, x_m \in R$ such that

$$(a_1 + a_{m+1} x_1)R + \cdots + (a_m + a_{m+1} x_m)R = R.$$

It is straightforward to see that, to verify the condition for stable range n given above, it is sufficient to do so in the critical case $m = n$. (It follows that if n is in the stable range for R, then so is any larger integer.) Also, although it appears that we should have referred to the above condition as R having "right" stable range n (since the definition was based on the use of right ideals), it has been shown later by Vaserstein [V$_1$: Th. 2] (and also by Warfield [Wa$_2$: Th. (1.6)]) that "right stable range n" and "left stable range n" are actually *equivalent* conditions. For this reason, we shall suppress any reference to side in referring to the stable range conditions defined in (8.1).

For readers familiar with Bass's big book [B$_{10}$], we should point out that there is a small discrepancy between the stable range notation used here and that used in [B$_{10}$]. What we called "stable range n" above corresponds to what Bass called SR$_{n+1}$ in his book. We believe the usage in (8.1) is now the standard one.

Bass's study of the stable range of rings was again motivated by the basic ideas of stability in the homotopy theory of vector bundles. His main results may be summarized in the fundamental theorem (8.2) below, which is to be thought of as the "K_1-analogue" of Corollary (2.6). Here, $\mathrm{GL}_n(R)$ denotes the $n \times n$ general linear

group over the ring R, and $\mathrm{E}_n(R)$ denotes its subgroup generated by the $n \times n$ elementary matrices $I_n + ae_{ij}$ ($i \neq j$, $a \in R$); $K_1(R)$ denotes the Whitehead group $\mathrm{GL}(R)/E(R)$ in algebraic K-theory, where $\mathrm{GL}(R)$ and $\mathrm{E}(R)$ are, respectively, the direct limits of the groups $\mathrm{GL}_r(R)$'s and $\mathrm{E}_r(R)$, taken over all positive integers r.

(8.2) Theorem. (See [B$_9$: (4.2), (11.1)] and [B$_{10}$: pp. 239-240].)

(1) *If a ring R has stable range n, then $\mathrm{GL}_n(R) \to K_1(R)$ is a surjective homomorphism. Moreover, for any $r \geq n+1$, $\mathrm{E}_r(R)$ is normal in $\mathrm{GL}_r(R)$, and $\mathrm{GL}_r(R)/\mathrm{E}_r(R) \to K_1(R)$ is an isomorphism.*

(2) *Let A be a commutative ring whose maximal ideal spectrum $\max(A)$ is a noetherian space of dimension d. Then, any module-finite A-algebra R has stable range $d+1$. In particular, the conclusions of (1) apply to such an algebra R for $n = d+1$.*

The second part of this theorem has been generalized to noncommutative noetherian rings R by Stafford; see [St]. In the case of commutative rings R, there has also been work done toward the removal of the noetherian hypothesis on the maximal ideal spectrum. For instance, Heitmann [Hei] has shown that, if R is a commutative ring of Krull dimension d, then R has stable range $d+2$ (and $d+1$ if R is a domain).

As one would perhaps expect, the case of stable range 1 has a special significance, not shared by the general case of stable range n. It is true, for instance, that having stable range 1 is a Morita invariant property of a ring, while, for $n > 1$, having stable range n is *not* a Morita invariant property [V$_1$, V$_2$, Wa$_2$]. The study of stable range has led to many new and unexpected results in the arithmetic of rings and the structure theory of modules. In the rest of this section (and the next), we shall give a survey of some of the interesting mathematics which resulted from this study, and which certainly would not have been possible without the pioneering work of Bass. We begin with the following useful observation that apparently first appeared in an unpublished note of Kaplansky [K$_3$].

(8.3) Proposition. *If a ring R has stable range 1, then R is Dedekind-finite; that is, $uv = 1 \in R$ implies that $vu = 1 \in R$.*

Proof (following [K$_3$]). Say $uv = 1 \in R$. Since $vR + (1-vu)R = R$, there exists $t \in R$ such that $w := v + (1-vu)t$ has a right inverse. Left-multiplying this equation by u, we get $uw = 1$. Therefore, w has also a left inverse. It follows that $w, u \in \mathrm{U}(R)$ (the group of units of R), and hence $vu = 1$. **QED**

For an alternative approach to this Proposition, see the beginning of §8 below. An immediate consequence of the Proposition is the following somewhat sharper formulation for the condition of stable range 1.

(8.4) Corollary. *A ring R has stable range 1 iff, whenever $aR + bR = R$, there exists $x \in R$ such that $a + bx \in \mathrm{U}(R)$ (the group of units of R).*

We recorded this corollary explicitly since the characterization for stable range 1 contained herein is often used as its *definition* in papers in the literature dealing with the stable range of rings.

In [B$_9$: (6.5)], Bass proved the following basic result:

(8.5) Theorem. *Any semilocal ring R has stable range* 1.

For a modern reader, this is a very natural result. One can first check (without too much difficulty) that a semisimple ring has stable range 1, and then deduce (8.5) from the observation that, in general, R has n in its stable range if $R/\mathrm{rad}(R)$ does. (For more details, see [La$_3$: pp. 313-314].) Conceptually, one can think of (8.5) as the "0-dimensional case" of part (2) of Theorem 8.2; in fact, in the case of commutative rings, R semilocal means that $\max(R)$ is *finite,* and therefore a 0-dimensional space.

Bass's study of stable range was quickly picked up by Estes and Ohm, who obtained various results in 1967 on stable range in the commutative case. The case of local rings shows already that a (commutative) ring R may have arbitrary Krull dimension, but still have 1 in its stable range. In [EO], Estes and Ohm constructed commutative Bézout domains of arbitrary dimension with 1 in their stable range. For other interesting examples of rings of stable range 1 and 2, see, for instance, the work of Heinzer [He], Jensen [Je], Menal-Moncasi [MM], and Warfield [Wa$_2$].

Warfield's paper [Wa$_2$] is a very important contribution to the study of the stable range of *noncommutative* rings. In this paper, he studied the endomorphism ring $E := \mathrm{End}_R(M)$ of a right module M over an arbitrary ring R, and sought characterizations for such a ring E to have stable range n (for a given integer n). Since Warfield's results lead to very interesting noncommutative cancellation theorems harkening back to Bass's results in §2, we shall give a quick exposition on the gist of [Wa$_2$] below. The first result of Warfield characterizes the stable range of $E := \mathrm{End}_R(M)$ in terms of a certain "substitution property" of M, as follows.

(8.6) Theorem. [Wa$_2$: (1.6)] *Let M be any right R-module with endomorphism ring $E := \mathrm{End}_R(M)$ (operating on the left on M). Then E has (right) stable range n iff M has the following "n-Substitution Property":*

$(S)_n$ *For any split epimorphism π from $T = M^n \oplus X$ to M, there exists a splitting φ such that $T = \varphi(M) \oplus Y \oplus X$, where $Y \subseteq M^n$.*

The proof of (8.6) consists of a fairly straightforward manipulation of the definition of stable range n.

(8.7) Remarks. (1) Note that if $(S)_n$ holds, then in the notation there we have

$$(8.8) \qquad\qquad \ker(\pi) \cong Y \oplus X, \quad\text{and}\quad M^n \cong Y \oplus M.$$

The first follows since both sides of the equation are direct complements of $\varphi(M)$ in T; the second follows similarly since both M^n and $Y \oplus \varphi(M)$ are direct complements of X in T (and $\varphi(M) \cong M$).

(2) Since the n-Substitution Property $(S)_n$ on M depends only on E by (8.6), it follows that if N is any module over any other ring with endomorphism ring isomorphic to E, then M satisfies $(S)_n$ iff N does.

To see the effect of (8.6) (and (8.7)) on module cancellations, we formulate the following slight improvement on Warfield's result in [Wa$_2$: (1.3)].

(8.9) Theorem. *Let M be any right R-module with endomorphism ring $E :=$ $\mathrm{End}_R(M)$ having stable range $n \geq 1$. For any right R-modules P and Q, the following are equivalent:*

(1) *For some module X, we have $P = M^{n-1} \oplus X$ and $M \oplus P \cong M \oplus Q$;*
(2) *For some modules X and Y, we have*

$$(8.10) \qquad P \cong M^{n-1} \oplus X, \quad Q \cong Y \oplus X, \quad \text{and} \quad M \oplus Y \cong M^n.$$

Proof. If (2) holds, then from (8.10):

$$M \oplus P \cong M \oplus M^{n-1} \oplus X \cong M^n \oplus X \cong M \oplus Y \oplus X \cong M \oplus Q.$$

Conversely, if (1) holds, let $T := M \oplus P = M \oplus M^{n-1} \oplus X$. Since $T \cong M \oplus Q$ has a split epimorphism π into M with $\ker(\pi) \cong Q$, the property $(S)_n$ for M implies the existence of a splitting φ such that $T = \varphi(M) \oplus Y \oplus X$, where $Y \subseteq M^n$. By (8.8), we have $M \oplus Y \cong M^n$, and $Q \cong \ker(\pi) \cong Y \oplus X$, as required in (2). (Note that the condition $Y \subseteq M^n$ from $(S)_n$ is never used in all of the arguments above.) **QED**

Note that, if we could have the implication

$$(8.11) \qquad\qquad M \oplus Y \cong M^n \Longrightarrow Y \cong M^{n-1},$$

then in (2) above, we would have been able to conclude that $P \cong Q$. While (8.11) does hold sometimes,[25] it cannot be counted on in the most general situation. To get a good cancellation theorem out of (8.9), we can, instead, impose a slightly stronger assumption on the module P, as in the following result.

(8.12) Warfield's Cancellation Theorem. [Wa$_2$: (1.2)] *Let M be any right R-module with endomorphism ring $E := \mathrm{End}_R(M)$ having stable range $n \geq 1$. If P contains a direct summand M^n, then for any R-module Q,*

$$M \oplus P \cong M \oplus Q \Longrightarrow P \cong Q.$$

Proof. Write $P = M^n \oplus W = M^{n-1} \oplus X$, where $X := M \oplus W$. If $M \oplus P \cong M \oplus Q$, we'll have the ismorphisms in (8.10) for some Y, and hence

$$Q \cong Y \oplus X \cong Y \oplus M \oplus W \cong M^n \oplus W = P,$$

as desired.   **QED**

In the special case when $E = \mathrm{End}_R(M)$ has stable range 1, the theorem above holds even *without any assumption* on P (or on Q); we shall come back to this point in §9.

In comparison with Bass's Cancellation Theorem (2.4)(2), the advantage of Warfield's (8.12) lies in the fact that there is neither a dimension assumption nor a noetherian assumption in its statement. Thus, (8.12) may be regarded as a "truly noncommutative" cancellation result. To see that (8.12) essentially retrieves the cancellability of a f.g. projective module M in the setting of (2.4)(2), we may first replace M there by R^m for some m, and then reduce to the case when $M = R$.

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[25]By a principle of A. Dress [La$_5$: (18.59)], (8.11) holds iff $E \oplus W \cong E^n$ (as right E-modules) implies $W \cong E^{n-1}$. Thus, (8.11) will hold if, say, stably free right modules over E are free and E has invariant basis number.

In this case, $E = \operatorname{End}_R(M) \cong R$. If R is module-finite over a commutative ring whose maximum spectrum is a noetherian space of dimension d, then by (8.2)(2), R has stable range $d+1$. So, as long as P has a free direct summand of rank $d+1$, (8.12) enables us to cancel off $M = R$ in (2.5).

In order to apply (8.12), one needs to know about the stable range of the endomorphism rings of modules. In this direction, Warfield has further generalized Bass's result (2.4)(2) by taking the module-finite A-algebra R there and considering finitely presented right modules M over R. In [Wa$_2$: (3.4)], Warfield showed that, if $\max(A)$ is noetherian of dimension d, then $\operatorname{End}_R(M)$ has $d+1$ in its stable range. Thus, one can apply (8.12) with $n = d+1$. Since M_R need no longer be a projective module, this leads to cancellation results well beyond the reach of (2.4)(2). Warfield's proof involved some new techniques, since Bass's methods did not apply to endomorphism rings.

§9. Rings of Stable Range One

To conclude our exposition, we shall consider in this section the case of rings of stable range 1. The crucial result here is that of Bass (Theorem 8.5), which states that semilocal rings have stable range 1. Again, as it turned out, this single result served as the fountain-head of many beautiful ideas to come; the survey in this section will show how much this result has stimulated subsequent research on the stable range of rings.

For the balance of this section, we shall only consider the stable range 1 case. First let us make the following useful observation.

(9.1) Remark. *Any module M_R satisfying the 1-Substitution Property $(S)_1$ in (8.6) is necessarily Dedekind-finite; that is, M is not isomorphic to a proper direct summand of itself.*

To see this, suppose there is an isomorphism $\pi : M \oplus W \to M$. Applying $(S)_1$ to π, we have a splitting φ for π (necessarily π^{-1}!) such that $M \oplus W = \varphi(M) \oplus Y \oplus W$ for some Y. But $\varphi(M) = T$, so W (as well as Y) must be zero, proving (9.1).

Note that (9.1) gives us another view of the fact (8.3) that a ring R of stable range 1 is Dedekind-finite. In fact, if R has (right) stable range 1, then, viewing R as $\operatorname{End}(R_R)$, (8.6) shows that R_R satisfies $(S)_1$, and hence (9.1) shows that R_R is Dedekind-finite. But this is the same as saying that its endomorphism ring R is Dedekind-finite (as a ring). This is a more conceptual, if somewhat longer, proof of (8.3).

Equipped now with the information in (9.1), let us take another look at the condition $(S)_1$ on M:

$(S)_1$ *For any split epimorphism π from $T = M \oplus X$ to M, there exists a splitting φ such that $T = \varphi(M) \oplus Y \oplus X$, where $Y \subseteq M$.*

Here, we have $M \cong Y \oplus M$ (as observed in (8.8)), so if $(S)_1$ holds for M, (9.1) implies that $Y = 0$, and hence the conclusion of $(S)_1$ simplifies to $T = \varphi(M) \oplus$

X. Thus, $(S)_1$ simply says that, if X and $\ker(\pi)$ both have direct complements isomorphic to M in a module T, then they have a *common* direct complement in T. We can therefore restate (8.6) in the case $n = 1$ as follows.

(9.2) Theorem. *The endomorphism ring* $E := \mathrm{End}_R(M)$ *of an R-module* M_R *has stable range* 1 *iff* M *has the following "Substitution Property": Whenever a right R-module* T *has direct decompositions* $T = M_i \oplus P_i$ *for* $i = 1, 2$ *where* $M_1 \cong M_2 \cong M$, *there exists a submodule* $C \subseteq T$ *such that* $T = C \oplus P_i$ *for* $i = 1, 2$.

Note that, in the above notation, if the submodule C exists, we have, in particular, $P_1 \cong T/C \cong P_2$. Therefore, we have the following consequences of (9.2).

(9.3) Corollary. (1) *If an R-module* M_R *has an endomorphism ring with* 1 *in its stable range, then* M *is "cancellable"; that is, for any right R-modules* P, Q,

$$M \oplus P \cong M \oplus Q \implies P \cong Q.$$

(2) *If a ring* R *has stable range* 1, *then* R_R *is cancellable, and hence so is any f.g. projective right (respectively, left) R-module.*

The result (9.3) was a cancellation theorem obtained earlier (ca. 1973) by E. G. Evans in [Ev]. We are reporting the results in the reverse chronological order here only because Warfield's result (9.2) is more general than Evans's, so it is logically more convenient to state (9.2) first and deduce (9.3) as its corollary.

To put things in the right historical perspective, we should also point out that what we called the "Substitution Property" in (9.2) had been considered as early as 1971 by L. Fuchs. In [Fu], apparently unaware of Bass's work on stable range, Fuchs obtained a different characterization of the Substitution Property, and proved the result (9.2) in the case $M_R = R$ (and some other cases), falling just short of proving (9.2) for *any* R-module M_R. Fuchs also considered the case of von Neumann regular rings R and sought conditions for R to have the Substitution Property (that is, to have stable range 1). He obtained the following result ([Fu: Cor. 1 and Th. 4]), which was also proved independently by Kaplansky [K$_3$: Th. 3].

(9.4) Theorem. *A von Neumann regular ring* R *has stable range* 1 *iff, for* $x, y \in R$, $xR \cong yR \implies R/xR \cong R/yR$.

Since any principal right ideal xR in a von Neumann regular ring R is a direct summand of the right module R_R (and R/xR is isomorphic to any direct complement of xR), the latter condition in Theorem 9.4 amounts to an "Internal Cancellation Property" of the module R_R; that is, isomorphic direct summands in R_R have isomorphic direct complements. Now by subsequent results of Ehrlich and Handelman on regular rings R ([Eh], [Ha$_1$]; see also [La$_4$: p. 242]), the module R_R has this property iff the ring R is *unit-regular*, in the sense that, for any $a \in R$, there exists a unit $u \in R$ such that $a = aua$. Therefore, (9.4) amounts to the fact that *a von Neumann regular ring* R *has stable range* 1 *iff* R *is unit-regular*. This statement seemed to have appeared explicitly for the first time in Henriksen's paper [Hen: Prop. 8].

While the above results on unit-regular rings are concerned mainly with the cancellation of f.g. *projective* modules, we should mention at least one case where (9.3)(1) applies more generally to *all* f.g. modules. This is a result due to Menal [Me], which deals with a certain subclass of unit-regular rings.

(9.5) Theorem. *Let R be a von Neumann regular ring all of whose primitive factor rings are artinian. Then for any f.g. projective right R-module M, $\mathrm{End}_R(M)$ has stable range 1. In particular, for any R-modules P and Q, $M \oplus P \cong M \oplus Q$ implies $P \cong Q$.*

Now let us come back to Bass's theorem (8.5). Combining this theorem with Evans's Cancellation Theorem (9.3), we see that *any module with a semilocal endomorphism ring is cancellable.* Another recent result of Facchini, Herbera, Levy and Vámos [FH: (2.1)] showed that, if P, Q are modules with semilocal endomorphism rings, then they have the "*n*-cancellation property" (for any integer $n \geq 1$); that is,

$$P^n \cong Q^n \Longrightarrow P \cong Q.$$

(For a detailed survey on *n*-cancellation, see [La$_6$].) These remarkable results give us strong motivation for finding classes of modules which have semilocal endomorphism rings. Now the problem of describing the endomorphism rings of specific kinds of modules has had a time-honored history, starting with the very famous Schur's Lemma:

(9.6) If M is a simple R-module, $\mathrm{End}_R(M)$ is a division (and hence semilocal) ring.

(9.7) Also well-known is the following classical generalization of Schur's Lemma: if M_R is a module of finite length, then $\mathrm{End}_R(M)$ is a semiprimary (and hence semilocal) ring; see, for instance, [La$_4$: Ex. (21.24)].

(9.8) Any strongly indecomposable module has, by definition, a local (and hence semilocal) endomorphism ring. Examples include, for instance, all indecomposable modules of finite length, and all injective (in fact all quasi-injective or pure-injective) indecomposable modules; see, e.g. [La$_5$: (3.52), Exer. (6.32)] and [F$_2$: (2.27)].

(9.9) Camps and Dicks [CD] showed in 1993 that *any artinian module has a semilocal endomorphism ring.*

(9.10) In 1995, Herbera and Shamsuddin [HS] generalized the Camps-Dicks result above by showing that, *if a module M has finite uniform dimension and co-uniform dimension,*[26] *then it has a semilocal endomorphism ring.* This class includes all linearly compact modules[27], and therefore all artinian modules. Thus, the Herbera-Shamsuddin result implies that linearly compact modules have semilocal endomorphism rings; this affirms an earlier conjecture of Faith. In the *commutative* case, a stronger conclusion is possible. In this case, any linearly compact module

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[26]A module M is said to have finite uniform dimension if there is a bound on the numbers n for which M contains a direct sum of n nonzero modules; dually, M is said to have finite co-uniform dimension if there is a bound on the numbers n for which M has a quotient that is a direct sum of n nonzero modules.

[27]An R-module M is said to be *linearly compact* if, for any submodules $N_i \subseteq M$ and any elements $m_i \in M$, any system of congruences $\{x \equiv m_i \pmod{N_i}\}$ that is finitely solvable is solvable.

M_R is algebraically compact (i.e. purely injective) according to Jensen and Lenzing [JL: p. 289]. Therefore, combining the Herbera-Shamsuddin result above with [JL: (7.5)], we see that M has a *semiperfect* endomorphism ring. This provides a nice connection back to the material of §5.

(9.11) A nonzero module is said to be *uniserial* if its submodules form a chain under inclusion. Such a module has clearly uniform dimension and co-uniform dimension both equal to 1. Thus, if $M := U_1 \oplus \cdots \oplus U_n$ where the U_i's are uniserial, then the "serial module" M has uniform and co-uniform dimension n, and hence it has a semilocal endomorphism ring by (9.10). For more information on the structure of $\mathrm{End}(U_i)$, $\mathrm{End}(M)$, and its applications to a (weak) Krull-Schmidt Theorem for serial modules, see Facchini's paper [F$_1$] and his recent book [F$_2$].

In view of the remarks made before (9.6), any modules of the type listed in (9.6)–(9.11) above have both the cancellation property and the n-cancellation property.

New results on stable range 1 are still being discovered today. Let us just mention a couple of more recent works. In [Ca], Canfell studied the stable range 1 condition from the viewpoint of completing diagrams of modules using automorphisms, and made connections to the notions of epi-projective and mono-injective modules. In [Ar], inspired by earlier results of Menal and Goodearl-Menal [GM], Ara proved the remarkable result that

(9.12) Theorem. *Any strongly π-regular ring (see §6) has stable range* 1.

In the commutative case, this amounts to the fact that any ring of Krull dimension 0 has stable range 1, which is close in spirit to part (2) of Bass's Theorem (8.2). In the general case, Ara's result is rather deep, and depends heavily on the use of noncommutative techniques. Combined with the earlier result of Armendariz, Fisher and Snider [AFS], (9.12) implies that any module with the "Fitting decomposition property" is cancellable: see (5.4) in [La$_6$].

In closing, we should also mention some interesting variations of the stable range one condition, due to Goodearl and Menal. In [GM], Goodearl and Menal considered the following two conditions for a ring E:

(A) $\forall \, a, b \in E$, $\ aE + bE = E \implies \exists \ u \in \mathrm{U}(E)$ such that $a + bu \in \mathrm{U}(E)$; and

(B) $\forall \, a, b \in E$, $\exists \, u \in \mathrm{U}(E)$ such that $a - u$ and $b - u^{-1} \in \mathrm{U}(E)$.

They showed that $(B) \Rightarrow (A)$, and obviously $(A) \Rightarrow$ stable range 1. They called condition (A) (right) "unit 1-stable range"; the unitary analogue of it has been exploited for C^*-algebras. Another variation, also due to Goodearl, is a *weakening* of stable range 1 condition. Extending Bass's definition, Goodearl defined a ring E to have the *power-substitution property* if, whenever $aE + bE = E$, there exists a positive integer n and a matrix $X \in \mathbb{M}_n(E)$ such that $a\,I_n + b\,X$ is a unit in $\mathbb{M}_n(E)$. (This property is also known to be left-right symmetric.) Clearly, if E has stable range 1, then it has the power-substitution property. The converse of this is, however, not true. For instance, it can be shown that the ring $\mathbb{Z}$ has the power-substitution property, but it certainly does not have stable range 1. The

raison d'être for the power-substitution property lies in the following important result of Goodearl on "power-cancellation" [Go$_2$: Cor. 4]:

(9.13) Theorem. *Let M, P, Q be right modules over an arbitrary ring R such that $M \oplus P \cong M \oplus Q$. If $E := \operatorname{End}_R(M)$ has the power-substitution property, then $P^n \cong Q^n$ for some $n \geq 1$.*

This result of Goodearl is, of course, an extension of (9.3)(1). In fact, in the case when the endomorphism ring E has stable range 1, the proof of Goodearl's result boils down to that of Evans, and gives the conclusion $P \cong Q$. The power of (9.13) stems from the fact that many different types of rings happen to have the power-substitution property; a good list of such rings is given in my survey [La$_6$: p. 34]. In §5 of this survey, there is also a summary of the many interesting results, due to Goodearl, Guralnick, and Levy-Wiegand, on the power-cancellation exponent n occurring in the statement of (9.13).

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